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Nathan Sidoli

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The Arithmetica *of Diophantus: A Complete Translation and Commentary* by Jean Christianidis and Jeffrey Oaks

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Reviewed by
Nathan Sidoli*
Waseda University
sidoli@waseda.jp

This book is not only an excellent translation and study of the extant Greek and Arabic sources for Diophantus' *Arithmetica*, it is also an important piece of scholarship in the history of premodern mathematics. The historiographic significance of this book comes from both its argument for the place of the *Arithmetica* in our understanding of the history of algebra and from the methodologies that the authors employ in order to make their case.

The *Arithmetica* was originally composed of 13 books, of which four are completely lost. There are six surviving books in Greek, preserved in 32 manuscripts, of which the earliest is from the 11th century and most are from the early modern period [18]. The treatise was translated into Arabic in the ninth century by Qusṭā ibn Lūqā, and four of the books of this translation survive in a single 12th-century manuscript, discovered by Fuat Sezgin in 1968 [22, 25]. In its various versions, the treatise was read by mathematicians as a work of algebra until the early modern period—as is made abundantly clear in many texts written by premodern mathematicians who studied the work. It was not until the 17th and 18th centuries that it began to be read as a work of number theory. Later, it was read as work of algebraic geometry—or at least as amenable to such a reading [Rashed and Houzel 2013]. Furthermore, historians of mathematics also read the *Arithmetica* as a work of algebra until fairly recently, when the contrary view was taken: that algebra proper was a purely Arabic-Islamic development, from which perspective the *Arithmetica* must then be regarded as a precursor in a more purely arithmetic tradition.

* Nathan Sidoli is professor of the history and philosophy of science at the School for International Liberal Studies, Waseda University, Tokyo. His research focuses on the Greek mathematical sciences and their further development in Arabic sources. He is author of *Thābit ibn Qurra's Restoration of Euclid's* Data (2018, with Yoichi Isahaya), and *The* Spherics *of Theodosios* (2023, with R. S. D. Thomas).

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This relatively recent view of the *Arithmetica* derives from the claim that algebra was an entirely new science developed in the classical Islamic period and originally canonized in al-Khwārizmī's *Concise Book on the Calculation of Restoration (*al-jabr*) and Confrontation (*al-muqābala*)*, usually simply called his *Algebra* [Rashed 2009]. Indeed, modern European languages derive their words for algebra from Latin versions of al-Khwārizmī's work, such as Robert of Chester's 12th-century translation titled *Liber algebrae et almucabola* [Karpinski 1915].

The emphasis on al-Khwārizmī's *Algebra* as the origin text of a new discipline that we call *algebra* derived from scholarship, particularly that of Roshdi Rashed [see, e.g., Rashed 1994], which correctly highlights the fact that al-Khwārizmī's work centers the algebraic *equation* and the various forms that this *equation* can take.[1] I believe that the view that algebra proper was originally an Arabic-Islamic development is still mainstream among historians of mathematics: we find such a view expressed, for example, in a fairly recent book by two eminent historians of mathematics—*Taming the Unknown: A History of Algebra from Antiquity to the Early Twentieth Century* [Katz and Parshall 2014, 32–173, esp. 137, 139, 158].

The authors of the book under review, however, along with their colleagues and coauthors, have been arguing against this view for more than a decade. Indeed, while it is true that al-Khwārizmī's *Algebra* centers the algebraic *equation*, this does not necessarily mean that algebraic problem-solving techniques were not already in use when al-Khwārizmī composed his text—as is, indeed, suggested by the fact that he does not even bother to explain the two most important algebraic operations, namely, *al-jabr* and *al-muqābala*. In this new book, we have the opportunity to read the culmination of the authors' work to reintroduce a reading of Diophantus as an algebraist to historians of mathematics. Their arguments for this can be divided into three main methodological approaches:

(1) extensive use of medieval mathematical scholarship, particularly that in Arabic;

[1] In this review, I shall write "*equation*" to refer to the principal relation of a premodern algebraic solution which the ancient and medieval authors discuss using expressions involving equality. In fact, there are many other relations stated in their texts that we can also interpret as equations but that they themselves do not explicitly call equations. Furthermore, I will occasionally also use the word "equation" in the normal sense, which should not be confused with the *equation* of a premodern algebraic solution.

(2) a close reading of the ancient and medieval sources themselves; and
(3) the use of a fairly restricted conceptual framework that Oaks has termed "premodern algebra".

1. Use of medieval scholarship

One of the keys to the reading presented in this book is the use of mathematical scholarship produced much later than Diophantus (*ca* AD 170 –370) and sometimes in languages other than Greek. Some of these later texts are commentaries on Diophantus, but most are original works of premodern mathematicians that advance their field in various ways, in the course of which they discuss the fundamental principles and practices of the field. In particular, this book makes extensive use of mathematicians working in Arabic who are discussing concepts and techniques that are also found in the *Arithmetica*. Although such a methodology should be applied carefully because later mathematicians may be introducing innovations not found in earlier sources, since the thought and approach of later medieval mathematical scholars' were much closer to that of the earlier sources than our own, their insight might well elude us if we only translated the ancient sources directly into our own idiom. I believe that the scholarship in this book presents a good example of how the comments, discussions, and mathematical procedures of later scholars who worked in a similar mathematical style and who were themselves thoroughly educated in the ancient methods may be used to shed light on previous developments.

2. Close reading and translation

Another important approach taken in this book involves a close reading of the Greek and Arabic sources themselves. This takes place in three stages of translation:

(1) Initially, the two texts are translated very literally, almost to the point of obscuring the mathematical sense from the modern reader. (This translation is even more literal than the French translations in Ver Eecke 1926 and Allard 1980.) For example, the Greek word «δύναμις» (lit. power, faculty) is rendered with "Power" and the Arabic term «مال» (lit. property, wealth) is simply transliterated as «Māl». This serves to highlight how alien these terms are to our thinking and to emphasize that we are dealing here with a specialized vocabulary. Furthermore, all of the many stylistic differences between the Greek and the Arabic are preserved, which makes it clear that we will need a conceptually coherent reading of the two superficially different

sources in order to understand how they both convey the same mathematical procedures and ideas.

(2) In the next stage, and especially in the commentary, part 3, an abbreviating symbolism is introduced that is directly modeled on that described by Diophantus in his introduction, found somewhat inconsistently employed in the Greek manuscript tradition of the text and much used by the Byzantine commentators on Diophantus' work. The abbreviations are based directly on the words used in (the English translation of) the two different texts, which makes the abbreviations themselves different for the Greek and the Arabic texts. For example, in *Arith.* 2.11 [304, 542, from Greek], "1 Power, 16 units lacking 18 Numbers...are Equal to 1 Power, 1 unit" is rendered

$$1\mathrm{P}\ 16\mathrm{u}\ \ell\ 8\mathrm{N} = 1\mathrm{P}\ 1\mathrm{u},$$

while in, *Arith.* 5.13 [388, 634, from Arabic], "nine *Māl*s and fifteen Things and nine units...are Equal to the nine *Māl*s and thirty units" becomes

$$9\mathrm{M}\ 15\mathrm{T}\ 9\mathrm{u}\ = 9\mathrm{M}\ 30\mathrm{u}.$$

When we see that in these types of expressions the Ns and Ts, or the Ps and Ms, are interchangeable, we can better understand how the Greek and Arabic texts, although their expressions are apparently so different, report the same types of mathematical procedures—at least in terms of the basic problem-solving techniques. Furthermore, it should be noted that, although these are symbols in the abstract semiotic sense, they are not symbols in the way that we usually employ the term in mathematics. That is, they do not represent mathematical objects or operations, but rather simply stand in for the words in the text. They serve as abbreviations for verbal expressions.

(3) In the final stage of transformation, which is standard for all studies of the *Arithmetica*, the expressions in the text are also translated into the sorts of algebraic expressions that we learn in school. This is done somewhat differently by different scholars, but, for example, the two previous abbreviations might be set respectively to

$$n^2 + 16 - 8n = n^2 + 1,$$

and

$$9t^2 + 15t + 9 = 9t^2 + 30.$$

Although we can, and often should, make such transformations in order to help ourselves to understand the mathematical situation better, the authors caution us to observe some crucial differences between the meanings conveyed by the modern symbolic expressions and the ancient abbreviations (here transliterated).

- In the first place, the modern use of either "+" or "−" implies an operation, whereas operations are actually expressed differently by Diophantus. Expressions like 1P 16u ℓ8N or 9M 15T 9u, however, are simply aggregates of numbers specified by their type (kind, or species) and enumerated by how many of each type are in the aggregate, some of which may be in deficit or owing.
- Second, the "N"s, "T"s, Ps, and "M"s of the abbreviations indicate the words in the text, which themselves name the sort of number that is being counted—a kind of unit, in the generic sense, of what is being enumerated. Our n or t, on the other hand, are symbols that indicate numbers—in the case of the *Arithmetica*, $n, t \in \mathbb{Q}$. That is, in "9M", the "9" counts a certain type, or species, of number which is called a *māl* or a *dunamis*, whereas in $9t^2$ the "9" is a scalar multiple of another number, namely, t^2. Of course mathematically, these give the same results, but conceptually they are quite different. Many of the detailed statements found in texts of premodern algebra cannot be understood if this distinction is not made.

Furthermore, the enunciations can also be translated into modern symbols, as has been done by all modern editors and translators of the text [67]. For example, using a notation modeled on that of Sesiano 1982, the enunciation of *Arith.* 2.11 [543] is given as:

$$x + a = \square$$

$$x + b = \square'$$

and that of *Arith.* 6.13 [633] as:

$$mx^2 + a = y + z$$

$$x^3 + y = \text{⧉}$$

$$x^3 + z = \text{⧉}'$$

where $\square$ is any rational square number, and ⧉ is any rational cube number. It should be clear these enunciations are much easier for us to grasp than the sometimes lengthy verbal expressions found in the text, but the authors caution that these enunciations use operations and results, as opposed to aggregates, involving a specialized naming system and statements of equality that are used to express the algebraic *equation*. Hence, the symbols that we use to summarize the enunciations are conceptually rather different from what is stated in the text, although mathematically they lead to the same results.

3. Premodern algebra

The final methodological component of the argument that the *Arithmetica* should be read as a text in algebra comes from framing this debate in terms of a historiographic framework developed by Oaks, which he calls *premodern algebra*. Oaks defined and discussed premodern algebra in six papers as a way of understanding the technical coherence of a certain type of mathematical practice reported in medieval Arabic, medieval Latin, and early modern European texts. Following, and somewhat concurrently, Christianidis has also argued in three papers, one written with Oaks, that Diophantus' *Arithmetica* is part of the tradition of premodern algebra [26–27].

The significance of defining premodern algebra as a historiographic category is that it helps us refine the scope of debates around the role of algebra in premodern mathematics. As is well known, historians of mathematics have engaged in long, and sometimes acrimonious, debates about the extent to which algebra and algebraic thinking have played a role in premodern mathematical texts and practices. One reason why there has been so little resolution to this debate may be that the terms involved—and in particular that of "algebra" and "algebraic"—are overly broad and understood differently by different scholars. For example, we might consider "algebra" from various perspectives to state equations involving unknown terms; to make statements of certain identities or mathematical laws; to use variables in order to represent numbers in computation and reasoning; to treat the sorts of structures that result from defining operations between the members of a set; and so on.

Since some of these ideas show up differently in different areas of premodern mathematics, while some of them are absent altogether, any attempt to make our terms more precise should be welcomed by historians of mathematics. One example of narrowing the focus to good effect can be found in Leo Corry's studies of what we would call the distributive law of multiplication over addition, $x \cdot (y + z) = x \cdot y + x \cdot y$, in medieval versions of Euclid's

Elements.[2] This work examines the various ways in which distributive properties are discussed and used by medieval mathematicians in texts that they regarded as dealing with geometry, arithmetic, and algebra. Another way to limit the scope of the discussion is to start out with the problem-solving methods that the medieval mathematical scholars themselves referred to as “algebra” and to describe the characteristics of this practice. Such an approach leads to the articulation of premodern algebra.

The use of premodern algebra as a historiographic tool is then necessarily, and intensionally, of limited scope. As a mathematical practice, it was a specialized technique for solving numerical problems and was not the only such technique available to premodern mathematicians. The solution of a numerical problem through premodern algebra involved the following essential features:

- The mathematician judiciously, but essentially arbitrarily, assigns terms involving specialized names that are drawn from a list of names designating a single unknown and its powers as well as the unit, to one or more of the variables of a numerical problem. Diophantus says that each of these technical names is “an element of the arithmetical theory” [276].
- Any operations implied in the statement of the problem are then discharged leading to the statement of an algebraic *equation* in a single unknown. (There are situations where another algebraic *equation* in another unknown is established as a sort of auxiliary [e.g., *Arith.* 4^{G}.16 and 4^{G}.17],[3] but in premodern algebra there is no technical vocabulary for stating *equations* involving more than one unknown, so that terminology for other unknowns had to be improvised or simply recycled.) Whereas there may be, and often are, variables involved in the statement of numerical problems, variables do not appear in the completed algebraic *equation*.
- The *equation* is then solved using a limited set of operations that are performed directly on the whole *equation* and are not

[2] See Corry 2021, which cites his earlier studies.

[3] The superscript “G” indicates that this book is numbered “four” in the Greek text. Because there are books numbered “four”, “five”, and “six” in both the Greek and Arabic texts, given that those in the Arabic are believed to be the actual fourth, fifth, and sixth books of the treatise, this notation is used for the later Greek books.

reduced to fundamental arithmetic operations. The two most important operations are those of *al-jabr* and *al-muqābala*, which are defined, although not so named, by Diophantus in the introduction to *Arithmetica* 1 as follows: “to add the lacking species on both sides, until the species on each side become extant” and “to subtract likes from likes on either side, until it results in one species equal to one species” [279].[4]

- Another operation may be defined at the beginning of *Arithemetica* book 4, which states:

  > If…we end up with one species of these species…Equals another species, we should divide everything by one of the lesser in degree of the two sides so that it results in one species Equals a number. [330]

- The solution of the *equation*, to which Diophantus himself gives relatively little attention, produces some numerical value for the unknown. This value is then used to solve the arithmetic problem that was originally posed.

This basic machinery is used in algebraic texts in Arabic, Latin, and early modern European vernacular languages.

With this limited description we can see that premodern algebra was not used to state identities, to relate variables or describe geometric lines (including curves), or to state physical laws or principles. It was certainly not used to study the structures that are determined by operations between the members of certain sets. The argument that the *Arithmetica* is a treatise in the tradition of premodern algebra is based on the claim that this highly structured problem-solving technique forms the core of Diophantus’ solutions to the arithmetic problems that he proposes.

To understand this assertion better, it may help to go through an example of the role of premodern algebra in a problem from Diophantus’ text. For this purpose, I will forego the abbreviations of the ancient text and use a notation that is a blend of that used by the authors and modern symbols for the algebraic problem-solving procedure, which I hope will make it easier for modern readers unfamiliar with the practice of premodern algebra to

[4] The ordering of *al-jabr* first and *al-muqābala* second, which is canonical in Arabic texts, probably derives, although not as a direct translation, from the way in which Diophantus expresses these two operations when he states them together at the same time [Sidoli 2022].

follow the discussion, but which is not mathematically different from the abbreviations given by the authors in their commentary.

The enunciation of *Arith.* 2.12 reads as follows: “To subtract the same number from two given numbers and to make each of the remainders a square” [305]. This is summarized by the authors in their commentary as follows [544]:

$$a - x = \square$$

$$b - x = \square'$$

where $a, b \in \mathbb{N}$ (rarely, assumed values may be proper parts of the form ⅟m where $m \in \mathbb{N}$, but not here). That is, where we have been assigned, or may choose, a and b arbitrarily, we want to find some other number—in this case $x \in \mathbb{Q}$—such that subtracting it from a and b individually, the results are both squares. Note that these are statements of operations and results.

This is, in fact, a problem of arithmetic, not premodern algebra, and the authors emphasize that although we can denote these conditions using symbolic equations, they are not expressed as *equations* by Diophantus and they are different from the *equation(s)* of the algebraic procedure that will lead to a resolution of the arithmetic problem.

Diophantus then sets $a := 9$ and $b := 21$, so that the arithmetical problem becomes:

$$9 - x = \square \tag{1}$$

$$21 - x = \square'. \tag{2}$$

It is important to point out that although we name the unknown number x in our summary, this is not yet named in the text; so we have not yet begun the part of the problem-solving procedure that uses premodern algebra. This begins in the next stage when we name the number sought.

The text then reads,

> whatever the square may be that I subtract from each of them, I assign (the sought-after number) to be the remainder; indeed, when this is subtracted, it leaves the square. [305]

That is, Diophantus is simply pointing out that he can *assign* the unknown number of the arithmetical problem, our x, to be whatever remains from subtracting the two different squares from the two given numbers, because

$$9 - (9 - \square) = \square$$

or

$$21 - (21 - \square') = \square'.$$

That is, considering equation (1) above, the simplest choice for the algebraic unknown, which we denote with n, would be to assign

$$x := 9 - n^2. \tag{3}$$

This assignment begins the premodern algebraic procedure.

Diophantus then points out that it is also necessary to subtract $9 - n^2$ from 21 so as to make a square because of equation (2) above, namely, $21 - (9 - n^2)$, must be a square, which leads to the first *equation* actually mentioned in the text. This is, in fact, what the authors call a *forthcoming equation*, namely,

$$[21 - (9 - n^2) = \square']$$
$$n^2 + 12 = \square'. \tag{4}$$

It is "forthcoming" because neither the square $\square'$ nor its side have been named in the terms of the algebraic problem-solving procedure—Diophantus' "arithmetic theory"—so that we are free to assign them in such a way as to render the problem amenable to solution. (The fact that $\square$ and $\square'$ are initially neither stated as given in the enunciation, nor named in the algebraic naming system, is the reason why they are denoted with this special notation, which was used by Tannery 1893–1895.)

Now, Diophantus uses a technique that the medieval Islamic mathematicians called *al-istiqrā'* [784–794] but that he himself neither names nor explains well, although he does give some hints as to how it functions. In essence, it involves choosing a side for the square $\square'$ in such a way that the *equation* thus resulting from equation (4) simplifies to two terms of consecutive powers (in practice usually eliminating the units or the terms of squared or higher powers). Here, Diophantus says,

> I form the square from 1 Number lacking so many units so that their square is greater than 12 units; indeed, in this manner, in either part one species will again be left equal to one species. [305]

That is, he signals his intention to eliminate one of the species, namely, the n^2, and to produce an *equation* lacking no units. Then, since $3^2 = 9 < 12$ and $4^2 = 16 > 12$, the simplest side that he can choose is $n - 4$, so that square $\square'$ becomes $(n - 4)(n - 4) = n^2 - 8n + 16$, which can be set equal to $n^2 + 12$ by equation (4). That is, as he states, "1 Power, 16 units lacking 8 Numbers...are equal to 1 Power, 12 units" [305]. This is the fully established algebraic *equation* in one unknown, namely,

$$n^2 + 16 - 8n = n^2 - 12.$$

He then simplifies this *equation* taking "likes from likes" so that

$$8n = 4$$

or

$$n = 4 \text{ 8ths.}$$

Diophantus then points out that this must be subtracted from 9, although he states neither the value of the unknown number, our x, nor those of the two squares, our $\square$ and $\square'$. (In fact, I believe that the authors' summary of the final step of this problem on p. 544 contains a typographic error: "$^{576}/_{64}$" should be "$^{560}/_{64}$".)

Since $9 = {}^{72}/_{8} = {}^{576}/_{64}$, it suffices to take away ${}^{16}/_{64}$ $\left(= ({}^{4}/_{8})^2 = n^2\right)$, which are the statements with which Diophantus concludes the problem. Although Diophantus finishes here, we may return from the algebraic result to the original assignment, equation (3), so as to flesh out the full solution to the arithmetic problem. In fact, the originally assigned unknown number (as opposed to the unknown of the algebraic *equation*), our x, is ${}^{576}/_{64} - {}^{16}/_{64} = {}^{560}/_{64}$, although Diophantus does not state this.

Hence, since $9 = {}^{576}/_{64}$ and $21 = {}^{1344}/_{64}$, the original arithmetic conditions are satisfied as follows:

$$\begin{aligned} {}^{576}\!/_{64} - {}^{560}\!/_{64} &= {}^{16}\!/_{64} = ({}^{1}\!/_{2})^2 = (2')(2'), \\ {}^{1344}\!/_{64} - {}^{560}\!/_{64} &= {}^{784}\!/_{64} = ({}^{7}\!/_{2})^2 = (3\,2')(3\,2'). \end{aligned}$$

We should remember, however, that the 9 and the 21 are also arbitrary, and various choices made in the course of the solution were motivated by these given numbers, so that there are any number of potential solutions to the original arithmetical problem. In this way, we see that the premodern algebraic procedure, which centers on the establishment and solution of a determinate *equation* involving specially named terms in a single unknown, serves as a mathematical tool for producing solutions to an indeterminate arithmetical problem.

With this as an introduction to the framework of premodern algebra, I now offer the following outline of the book under review.

4. Outline

Part 1. Introduction [1–271]

Diophantus and his work [3–25] A discussion of our almost complete lack of knowledge of Diophantus' life, situated sometime between *ca* AD 170 and *ca* AD 370, is followed by a short description of his known works, a list of the manuscripts of the *Arithmetica* and their stemmata (from Tannery 1893–1895; Allard 1980; and Acerbi 2011), and an overview of all editions and previous translations of the text.

Numbers, problem solving, and algebra [26–79] This introduction to premodern algebra, organized topically, is crucial for readers unfamiliar with this historiographic generalization.

(1) A historiographic sketch of the history of scholarship on premodern algebra, distinguishing it from other premodern methods of solving for unknown values, such as those found in Babylonian or Indian sources [Høyrup 2002; Plofker 2009, 191–196].

(2) An introduction to the types of numbers used in premodern arithmetics, which form the fundamental conceptual background for understanding the mononomials and polynomials of premodern algebra *as numbers*. A common category of number in premodern sources used to be called a "complex" or compound number. Numbers such as 13 12′ 45′ hours (in value, $13 + 1/12 + 1/45$), 23 51 20 degrees (in value, $23 + 51/60 + 20/60^2$), 173 less 8′ degrees (in value, $173 - 1/8$), 8 *livres* 8 *sols* 8 *deniers* (in value, $8 + 8/20 + 8/12 \cdot 1/20$), and so on are all examples of compound numbers.[5] Each of the numerals in these expressions states, or enumerates, a different kind of thing—a number of hours, a proper part, a number of degrees, of sexagesimal parts, of a certain unit of currency, and so on. The overall value referred to in these expressions is compounded from all of the enumerated terms. The ways of computing with such compound numbers were discussed in arithmetic texts and commentaries throughout the ancient, medieval, and early modern periods. The crucial point of this section is that the polynomials of premodern algebra are conceptually the same as these compound numbers. An expression like "1 Power 64 units lacking 16 Numbers" denotes a compound number made up of one of those things whose units we are calling "Power", 64 of those things whose units are the unit (but we might use other terms), and in deficit by 16 of those things whose units we are calling "Number". Hence, the words "unit", "Number", and "Power" are not themselves numbers like the n in our expression $n^2 + 64 = 16n$, which has the same mathematical structure as "1 Power 64 units lacking 16 Numbers", they are simply the designations of the type of number enumerated by the preceding numerical value. Actual calculations could be carried

[5] See *Historia Mathematica* 59 (2022) for recent historical work on such numbers.

out using a number of different methods, but these are usually not covered in texts on premodern algebra.

(3) A general discussion of numerical problem-solving shows that premodern algebra was just one of a number of different techniques that could be utilized to solve such problems.

(4) The use of premodern algebra as one possible type of solution to a numerical, or arithmetical, problem is explained. (See also (6) below.)

(5) Monomials, polynomials, and the *equation* in premodern algebra are defined. In particular, it is stressed that a polynomial in premodern algebra does not contain any operations but is rather an aggregate made up of various types of numbers such as simple numbers or units, unknown Numbers or Things (n), Powers or *Māl*s (n^2), and so on (n^3, n^4, and so on), any (aggregate) of which may be in deficit (*lacking*, *less*, or $-$), and which aggregate is itself a compound number in the sense discussed above. A *forthcoming equation* is an important subcategory of the algebraic *equation*, namely, one in which a monomial or polynomial is asserted to be equal to a square, or cube, that has, as of yet, no terms designated in the algebraic naming system.

(6) The procedure of an algebraic solution itself is described in four stages:

   (i) the conditions of the arithmetical problem are converted into an algebraic *equation* by assigning names to the unknown terms of original problem and discharging any operations implied by the conditions of the problem;

   (ii) the *equation* is simplified to a standard form using the operations of *al-jabr* and *al-muqābala*, and for which Diophantus used expressions like "Let a common, the lacking, be set out" and "Let the same be subtracted from the same";

   (iii) the simplified *equation* is solved for the single unknown;

   (iv) the unknowns of the original arithmetical problem are computed, using the now known value named in the algebraic assignment and computed using the algebraic *equation*.

(7) The next section treats the difference between the enunciation and the algebraic *equation*, pointing out that while the problems are often indeterminate, the *equations* almost never are, and discusses the various techniques for assigning names to the unknowns.

(8) A final section discusses the notation used in the Greek books of the *Arithmetica* and argues that it is essentially a way of abbreviating the expressions in the text, so that there is no fundamental conceptual

difference, between the abbreviated symbolism of the Greek text and the fully rhetorical expressions of Qusṭā's translation. This implies that Nesselmann's tripartite division of algebra into rhetorical, syncopated, and symbolic [1842] does not help us to understand the practices of premodern algebra.

History [80–230] This section gives a historical account of the *Arithmetica* as a text of premodern algebra, describing both the mathematical contexts in which the text was produced and read along with particular usages and readings of the text found in other texts of premodern algebra. (This is the longest section of the introduction and I mention only some highlights.) The authors revive the now unpopular view that Hipparchus wrote a text on algebra in the 2nd century BC, which is asserted in the Islamic bibliographic literature. They also discuss three algebraic problems on the 2nd-century AD papyrus P.Mich.inv. 620, which was probably written before the *Arithmetica*. Such discussions make it clear that premodern algebra was a mathematical practice predating Diophantus' work.

After a treatment of the ancient accounts of Diophantus and his work, including some problems solved using "Diophantine numbers", the authors discuss premodern algebra in the classical Islamic period, before the *Arithmetica* was translated into Arabic. This provides an introduction to the context as well as the Arabic technical terminology of the translation by Qusṭā ibn Lūqā. This is followed by a lengthy survey of the many mathematicians of the medieval Islamic period who discussed the *Arithmetica* or solved problems that derived from the text.

The authors then turn to scholarship on Diophantus in the Byzantine Empire, which produced the most important Greek sources for the text as well as important comments and scholia on them. This is followed by a review of the early modern discovery of the text, its translation into Latin, its Greek *editio princeps*, and the integration of Diophantus' work into the early modern tradition of premodern algebra, the practice of which predated any study of *Arithmetica*.

The section concludes with a brief account of the beginning of modern algebra, adumbrated in the work of François Viète. The examples of the transmission of the *Arithmetica* in both the ninth-century Islamic civilization and the European Renaissance shows that premodern algebra was a mathematical practice that could be, and was, transmitted independently of canonical texts, probably through the direct oral transmission of mathematical practices.

Throughout the whole section, particular attention is paid to the great variety of terms used for the "unit" and the unknown and its powers (n, n^2, n^3, and so on) in the ancient and medieval sources.

Structure and language of the Arithmetica [231–266] In the *Arithmetica*, the four stages of an algebraic solution are nestled inside the structure of the overall arithmetical problem, which is itself modeled on that of propositions in Euclid's *Elements*. The technical terminology used in both the Greek and Arabic texts is defined and discussed.

Didactic aspect of the Arithmetica [267–271] This section argues that the *Arithmetica* shows evidence of Diophantus' overall concern with the didactic presentation of material by focusing on:

(1) explicit statements that he makes in this regard,
(2) various explanations of assignments and procedures that he sprinkles throughout his problems,
(3) the occasional false start (which the authors call a "constructive dead-end") that he includes in the text before giving the successful solution, and
(4) the over arrangement of the problems, which allows the reader to learn various techniques of solution as a sort of "toolbox" that can be used in later problems.

Part 2. Translation [273–506]

The English translation begins with the first three Greek books, 1–3, followed by books 4–7 from the Arabic, and concluding with the Greek books 4^G, 5^G, and 6^G. The translation is highly systematic and literal. The technical terms are rendered with English words having the literal meanings of those terms (and a transliteration), and the numbers are even translated with digits for the Greek books and with words for the Arabic books, just as we find in the original sources. While I agree that this is the best choice, given that the technical terminology and numerals are not homogenized across the Greek and Arabic books, I suspect that many readers will take some time to adjust to these differences. A deliberate use of capitalization and italics allows the translation to remain literal, while still highlighting the way that the naming practice of premodern algebra, what Diophantus calls the "arithmetic theory", is employed in the treatise. The translation from the Greek text is enumerated in the margin with the pagination of Tannery's edition [1893–1895]; that from the Arabic is enumerated by the line numbers of Sesiano's edition [1982] at the start of each paragraph of the English translation.

Part 3. Commentary [507–778]

A discussion of the abbreviating notation is followed by a symbolic summary of each problem. The enunciation is provided with a statement in modern symbols that the authors stress is not faithful to the intent of the original but that is nevertheless useful to modern readers for understanding the mathematical idea of the problem. This is then followed by a symbolic treatment, using the notation developed by the authors that is simply an abbreviation of what the text actually says of the assignments, operations, and finally the *equation(s)* that can be used to produce the numbers that will solve the problem. The symbols used for the algebraic part of each problem are the first letters of the technical terms in Greek and Arabic, so that the principal symbols, although not their mathematical meanings, are different for the Greek and the Arabic books. For some problems the authors provide mathematical, historical, and philological remarks, including some translations of medieval scholarship on the text.

Part 4. Appendices [779–840]

Appendix 1 [781–783] A translation of four missing problems of the Greek book 5^{G}, reconstructed by E. S. Stamatis.

Appendix 2 [784–798] A discussion of some techniques for solving indeterminate problems in premodern algebra:

(1) The first section deals with the technique called *al-istiqrā'* by the medieval Islamic algebraists, which involves assigning an as-of-yet unnamed square (or cube) in such a way that some of the terms, such as the squares or the units, drop out of the *equation* that is thus established. There follows a list of all of the problems that involve forthcoming *equations* solved by *al-istiqrā'* in the *Arithmetica*, along with the assignment of the square, the cube, or its side that effects the requisite elimination.

(2) The second section treats a number of techniques that are used to handle simultaneous forthcoming *equations*, of which one side is an as-of-yet unnamed square. These techniques are discussed and classified, and a list of all the problems that employ them is provided, stating the forthcoming *equations*, the approach used, and the key to the solution.

Appendix 3 [799–811] Glossary of the English terms used to translate the technical terminology of both the Greek and Arabic texts.

Appendix 4 [812–840] Conspectus of all the problems using modern symbols (as seen in the example discussed above), which helps the modern reader to understand at a glance what the problems propose.

Bibliography and Index [841–876]

5. Conclusion

Following this overview of the entire book, I should perhaps raise some minor criticisms. In a book of this size the careful reader is bound to find a number of typographic errors. (I doubt that the publisher would be able to find a copyeditor who could read such a text with the care necessary to catch all of them.) Such typos are especially found in the abbreviations that the authors use to summarize the actual steps of the mathematical argument. For example, there are two such typos on p. 785 in lines 22 and 30, which one will notice as soon as one converts the abbreviations into the modern symbols with which we are more familiar. Furthermore, there is some confusion in the references for papyri. For example, when the well-known P.Mich.inv. 620 (P.Mich. III.144, now available online in high quality color images) is introduced on p. 70, it is referenced as "Problem III.iv, edited and translated in (Winter 1939, 39, 44–45)". Moreover, when the authors discuss this papyrus, now named more fully on pp. 102–110 and give a valuable interpretation of its mathematics as belonging to premodern algebra, they continue to credit J. G. Winter with the editorial work on P.Mich.inv. 620. Winter, however, was the editor of the whole of volume 3 of the Michigan Papyri series, while the editor of the papyri concerning mathematical subjects in this volume, including the astral sciences, was F. E. Robbins, with each such article being signed, somewhat cryptically, "F.E.R".

These minor issues asides, this book represents a major accomplishment in the historiography of premodern mathematics. It presents a comprehensive and coherent reading of both the Greek and the Arabic texts of Diophantus' *Arithmetica* that allows us to give a systematic account of Diophantus' mathematical approach and problem-solving techniques that remains consistently close to what we read in the sources themselves. It helps us to understand the *Arithmetica* as a book in a long tradition of premodern algebra, which although practiced by some seems not to have become mainstream among philosophically trained Greek mathematicians but eventually developed into a focused area of study at the hands of the numerous algebraists of classical and medieval Islamic societies. Finally, it provides a valuable example of the ways in which, for the premodern period, much later texts that are sometimes produced in a different language can be used to shed light on earlier mathematical practices and ideas. I believe that this book makes a valuable contribution to our understanding of premodern mathematics and to the history of mathematics more generally.

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