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Article abstract

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# DYNAMIC OPTIMAL FUTURES HEDGING WITH JUMP RISK AND STOCHASTIC INTEREST RATES

**by Nabil Chahdoura, Minh Chau To, and Pierre Laroche**

**ABSTRACT**

In this paper, we derive dynamic optimal futures hedge ratios when the relative change in the spot price follows a Poisson jump-diffusion process, and when basis and marking-to-market create additional risks. We show that our hedge ratios could be more efficient than the regression hedge ratio, and those obtained by Chang, Chang and Fang (1996a, 1996b), either because of a different model specification, or because of a different process specification. We apply our model to the West Texas Intermediate crude oil contract quoted on the NYMEX, and we show that our hedge ratios are more efficient than the other ratios under the shorter hedging horizons.

**RÉSUMÉ**

*Dans cet article, nous développons des ratios de couverture dans un contexte dynamique, par l'utilisation des contrats à terme boursiers, lorsque le prix au comptant suit un processus de diffusion avec sauts de Poisson, et lorsque la base et le règlement quotidien créent des risques additionnels. Nous montrons que nos ratios de couverture pourraient être plus efficients que celui induit par la régression, et ceux obtenus par Chang, Chang et Fang (1996a, 1996b), soit à cause d'une modélisation différente, soit à cause de la spécification d'un différent processus de diffusion. Nous avons appliqué notre modèle au contrat sur le pétrole brut West Texas Intermediate coté sur le NYMEX, et montré que nos ratios de couverture sont plus efficients que les autres ratios dans les horizons de couverture courte.*

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## INTRODUCTION

Corporate risk hedging is an important managerial function, leading to the emergence of a large and diversified body of literature on the subject since the early '60s.[1] The early models of risk hedging with futures assumed that financial assets returns are uncertain variables and therefore derive uniperiodic hedging ratios which optimize some given corporate or individual investor's objective, such as the minimization of the investor's portfolio variance.

More generally, however, returns on financial assets can be assumed to follow stochastic processes with a normal or log-normal volatility structure. Furthermore, because of the uncertain arrival of important information which strongly impacts the financial assets returns, they can also be assumed to show jump risk. Brennan & Schwartz (1990) and Chan (1992) have showed that basis (defined as the spot price minus or divided by the future price) risk is significant, and that hedging ratios that do not take into account the jump risk could be inefficient.

Chang, Chang and Fang (1996a) proposed intertemporal hedge ratios in a dynamic setting where the spot and futures returns follow jump-diffusion processes, and where the jump component is a standard homogeneous Poisson process. Assuming no marking-to-market of the futures (no forward-futures bias) and a constant correlation between the spot and futures prices (no basis risk), the hedge ratio (hereafter $h_1$) obtained by Chang, Chang and Fang by minimizing the hedge portfolio variance does not differ from the traditional or regression hedge ratio, when the jump components were ignored:[2]

$$h_1 = \frac{\sigma_{SF} + j\sigma_{xy}}{\sigma_F^2 + j\sigma_y^2} \qquad (1)$$

where:

- $\sigma_{SF}$ is the covariance between the relative change in the spot price $S$ and the relative change in the futures price $F$;
- $\sigma_{xy}$ is the covariance between $x$, the gross jump size of the spot price, and $y$ the gross jump size of the futures price;
- $j$ is the arrival frequency parameter of the Poisson process which is common to both the spot and futures price processes;
- $\sigma_F^2$ is the variance of the return on the futures price, $F$; and

$\sigma_y^2$ is the variance of the gross jump size of the futures price $y$.

When a net cost-of-carry (which could be negative) is introduced as an Orstein-Uhlenbeck process, the relative basis risk is also shown to follow a jump-diffusion process similar to those of the spot and futures prices. Because of the basis risk, and if the spot return and the net cost-of-carry (and thus, the relative basis) are positively correlated, the hedge ratio in this case (hereafter $h_2$) is smaller than any at all times before maturity.[3]

$$h_2 = \frac{\sigma_S^2 + \delta_{Si}\sigma_S\sigma_b + j\left\{\sigma_x^2 + \delta_{xw}\sigma_x\sigma_w(T-t)^{1/2}\right\}}{\sigma_F^2 + j\sigma_y^2} \qquad (2)$$

where:

$\sigma_S^2$ is the variance of the relative change in the spot price $S$;

$\delta_{Si}$ is the correlation between the spot return, and $i$, the rate of the net-cost-of-carry;

$\delta_{xw}$ is the correlation between $x$, the gross jump size of the spot price, and $w$, the gross jump size of the rate of change of the relative basis defined as $B = S/F$, the ratio of the spot price $S$ to the futures price $F$;

$\sigma_b$ is the standard deviation of the rate of change of the relative basis $B$;

$\sigma_w$ is the standard deviation of $w$, the gross jump size of the rate of change of the relative basis, together with a Poisson process;

$\sigma_x^2$ is the variance of $x$, the gross jump size of the spot price;

$T$ is the maturity date of the futures; and

all other variables are as defined earlier.

In addition, since the futures volatility increases with the relative basis volatility, the future volatility and the hedge ratio are negatively correlated. While this hedge ratio is richer than the traditional one by accounting for the basis risk, its application to a better understanding of hedging practices is hampered by the assumptions of deterministic jump arrival shifts, and of the same standard homogeneous Poisson process for the spot price, the futures price and the relative basis. Moreover, since by definition the relative basis is the ratio of the spot price to the futures price, the latter is endogenous to the model, and therefore it is unclear whether the introduction of basis risk is only a simple substitution for the exogenous futures price risk in the former model. This is especially true if one assumes that the Poisson process affecting the spot price is not the same as that affecting the futures price, either precisely because of a stochastic net cost-of-carry or a stochastic interest rate, or because of the influences of macroeconomic variables.[4]

Furthermore, minimizing the hedged portfolio variance, as Chang, Chang and Fang (1996a) did, does not recognize the investor's objective of maximizing expected return,[5] and the daily marking-to-market in futures trading imposes an additional risk when the interest rate (a component of the net cost-of-carry) is stochastic.[6] To address these issues, Chang, Chang and Fang (1996b) proposed intertemporal hedge ratios (hereafter $h_{3F}$ and $h_{3B}$) in a continuous time framework, where the (absolute) changes in spot and futures prices follow arithmetic Brownian processes, and where the risk-free interest rate follows a mean reverting square root process (see Cox, Ingersoll and Ross (1985).[7] The existence of two hedge ratios arises from the need to hedge against settlement (marking-to-market) risk by holding $h_{3B}$ risk-free bonds, along with $h_{3F}$ futures contracts to cover against the spot price risk. The two hedge ratios are obtained by assuming that the investor is endowed with a time-additive, state-independent Von Neumann-Morgenstern utility function, $U[C(t),t]$, which is strictly increasing and concave in consumption $C$. Chang, Chang and Fang (1996b) were able to show that their hedge ratios are generalizations of those obtained by Ho (1984), Stulz (1984), and Adler & Detemple (1988) when $U[C(t),t]$ is logarithmic and interest rates are not stochastic. The mishedging by one-period hedge ratios, illustrated by numerical examples based on the S&P 500 futures, is shown to be significant, especially for shorter hedge durations. The futures-forward differential caused by the covariances between interest rates and spot and futures prices implies settlement risk which is non-trivial and which should be hedged against. Chang, Chang and Fang (1996b) also noted that since the futures-forward bias is asset-specific, little could be generalized from any numerical example based on a given commodity.

The purpose of this paper is to extend the results of Chang, Chang and Fang (1996b) to stochastic processes of the spot and futures returns which take into account jump risk. We are able to show that our hedge ratios are more efficient than $h_{3F}$ and $h_{3B}$

proposed by Chang, Chang and Fang (1996b), especially for the more volatile spot price.

In the next section, the model will be discussed, and the proposed hedge ratios will be contrasted with benchmarks $h_{3F}$ and $h_{3B}$.[8] Then, a numerical example based on the West Texas Intermediate crude oil will be presented, and again, the results will be compared with those obtained by applying the hedge ratios $h_{3F}$ and $h_{3B}$. The last section concludes the paper.

## THE MODEL

We assume the following:

**A1.** The markets are perfect, and trading in both the spot and futures markets occurs continuously.

**A2.** The spot price S follows a mixed Poisson-diffusion stochastic process described by the following stochastic differential equation:

$$\frac{dS}{S} = \left[\alpha_S - \lambda_S k_S\right] dt + \sigma_S \, dz_S + dq_S \tag{3}$$

where:

$\alpha_S$ is the instantaneous expected spot return;

$\lambda_S$ is the expected number of jumps (or probability of a jump) in the spot price at each point in time;

$k_s$ is the expected size of the jumps in spot price, measured by the relative change in the spot price;

$\sigma_S$ is the instantaneous standard deviation of the spot return, conditional on the absence of a jump in the spot price;

$dz_S$ is the increment in the standard Wiener process specific to the spot return;

$dq_S$ is the standard Poisson process which describes the arrival model of information capable of inducing a jump in the spot price, $S$;

$i$ is the default-free interest rate.

It is assumed that $\text{cov}(dz_S, dq_S) = \sigma_{S,qS}$, and that the margin requirement on cash trading is 100%.

**A3.** The investor has free and unlimited access to the futures market, where no initial margin is required, and marking-to-market is continuous. The futures price $F$ follows a mixed Poisson-diffusion stochastic process described by the following stochastic differential equation:

$$\frac{dF}{F} = \left[\alpha_F - \lambda_F k_F\right] dt + \sigma_F \, dz_F + dq_F \tag{4}$$

where:

- $\alpha_F$ is the instantaneous expected futures return;
- $\lambda_F$ is the expected number of jumps (or probability of a jump) in the futures price at each point in time;
- $k_F$ is the expected size of the jumps in the futures price, measured by the relative change in the futures price;
- $\sigma_F$ is the instantaneous standard deviation of the futures return, conditional on the absence of a jump in the futures price;
- $dz_F$ is the increment in the standard Wiener process specific to the futures return;
- $dq_F$ is the standard Poisson process which describes the arrival model of information capable of inducing a jump in the futures price, $F$;

and all other variables are as defined earlier.

It is assumed that $\text{cov}(dz_F, dz_S) = \sigma_{SF}$; $\text{cov}(dz_F, dq_F) = \sigma_{F,qF}$; $\text{cov}(dz_F, dq_S) = \sigma_{F,qS}$; $\text{cov}(dz_S, dq_F) = \sigma_{S,qF}$; and $\text{cov}(dq_S, dq_F) = \sigma_{qS,qF}$. Any covariance term above which involves a Poisson process is conditional upon the absence of jumps in that process.

**A4.** The investor has free and unlimited access to continuous lending and borrowing at the default-free rate $i$, which is stochastic and follows a Cox, Ingersoll and Ross (1985) square root process:[9]

$$di = k(\alpha_i - i)dt + \sigma_i \sqrt{i} \, dz_i \tag{5}$$

where:

- $\alpha_i$ is the instantaneous long run expected interest rate;
- $k$ is the speed parameter of reversion towards the long-run expected interest rate;

$\sigma_i$ is the instantaneous standard deviation of the interest rate;

$dz_i$ is the increment in the standard Wiener process specific to the interest rate $i$; and all other variables are as defined earlier.

It is assumed that $\text{cov}(di, dz_S) = \sigma_{Si}$; $\text{cov}(di, dz_F) = \sigma_{Fi}$; $\text{cov}(dz_i, dq_S) = \sigma_{i,qS}$; and $\text{cov}(dz_i, dq_F) = \sigma_{i,qF}$. Again, any of the above covariance terms which involves a Poisson process is conditional upon the absence of jumps in that process.

To hedge against basis risk and settlement (marking-to-market) risk, an interest-rate-sensitive instrument is needed, namely a default-free bond. If $B(i,t,T)$ is the price at time $t$ of such a bond, which pays one dollar at maturity date $T$, when the interest rate is $i$, then Cox, Ingersoll and Ross (1985) have shown that it is expressed as:

$$B(i,t,T) = D(t,T)\, e^{-i\,G(t,T)} \tag{6}$$

where:

$$D(t,T) = \left[\frac{2\tau e^{(k+\pi+\tau)(T-t)/2}}{(k+\pi+\tau)\,(e^{\tau(T-t)}-1)+2\tau}\right]^{2k\alpha_i/\sigma_i^2}, \tag{7}$$

$$G(t,T) = \frac{2(e^{\tau(T-t)}-1)}{(k+\pi+\tau)(e^{\tau(T-t)}-1)+2\tau}, \tag{8}$$

$$\tau = \left[(k+\pi)^2 + 2\sigma_i^2\right]^{1/2}, \tag{9}$$

and $\pi$ is the market risk parameter of interest rates.[10]

Using Itô's lemma, the dynamics of the bond price is given by the following stochastic differential equation:

$$\frac{dB}{B} = i(1-\pi G)\,dt - G\sqrt{i}\,\sigma_i\,dz_i\,. \tag{10}$$

**A5.** The investor is endowed with a time-additive Von Neumann-Morgenstern utility function, $U[C(t),t]$, which is strictly increasing and concave in consumption $C(t)$. The investor's value function is therefore:

$$J(W,S,i,t)=Max\int_{t_0}^{T}E\left\{U[C(t),t]dt,Q[W(T),T]\right\} \tag{11}$$

subject to:

$$dW=x\,dS+y\,dF+z\,dB-C\,dt \tag{12}$$

where:

$E$ is the expectation operator; $t_0$ is the current time;

$Q$ is the bequest function which is strictly increasing and concave in $W$, the terminal wealth at the end of the investor's horizon,

$T$ (which coincides with the maturity date of the futures); and

$x$, $y$ and z are respectively the holdings in spot contracts, futures contracts and default-free bonds.

Using the expression for $dS$, $dF$ and $dB$ in (3), (4) and (10) respectively, the constraint in (12) which describes the change in the investor's wealth at each point in time can be rewritten as:

$$\begin{aligned} dW=&\left[x(\alpha_S-\lambda_S k_S)S+y(\alpha_F-\lambda_F k_F)F+zi(1-\pi G)B-C\right]dt \\ &+x\sigma_S S\,dz_S+y\sigma_F F\,dz_F-z\,BG\sqrt{i}\sigma_i dz_i+S\,dq_S+F\,dq_F \end{aligned} \tag{13}$$

The solution to the investor's program (11)-(12) is to find $x$, $y$, $z$ and $C$ such that J is achieved. The technical tool for this purpose is stochastic dynamic programming, which implies the following Bellman equation:

$$0=\underset{(x,y,z,C)}{Max}\left\{\begin{matrix} u[C(t),t]+J_\zeta \\ +J_w\,D_w+J_S\,D_S+J_i\,D_i \\ +\tfrac{1}{2}\left[J_{ww}\,V_w+J_{SS}\,V_S+J_{ii}\,V_i\right] \\ +J_{wS}\,Cov_{wS}+J_{wi}\,Cov_{wi}+J_{Si}\,Cov_{Si} \end{matrix}\right\} \tag{14}$$

where subscripts to $J$ designate first (single subscript) and second (double subscript) partial derivatives of $J$ with respect to the subscripted variables; $\zeta = T - t$ is the time to the end of the investment horizon (futures maturity date); subscripts to $D$ designate the drift term relative to the subscripted variables; subscripts to $V$ designate the unconditional variance of the subscripted variables; and subscripts to $Cov$ designate the unconditional covariance between the subscripted variables.

More specifically, (14) can be rewritten as:

$$0 = \underset{(x,y,z,C)}{Max}\left[\begin{array}{c} u[C(t),t] + J_\tau \\ +J_w\left[xS(\alpha_S - \lambda_S k_S) + yF(\alpha_F - \lambda_F k_F) + zBi(1-\pi G) - C\right] \\ +J_S(\alpha_S - \lambda_S k_S)S + J_i k(\alpha_i - i) \\ +\frac{1}{2}J_{ww}\left(x^2S^2\sigma_S'^2 + y^2F^2\sigma_F'^2 + z^2B^2G^2\sigma_i^2 i\right) \\ +J_{ww}\left(xy\sigma_{SF}'SF - xz\sigma_{Si}'SBG\sqrt{i} - yz\sigma_{Fi}'FBG\sqrt{i}\right) \\ +\frac{1}{2}J_{SS}S^2\sigma_S'^2 + \frac{1}{2}J_{ii}i\sigma_i^2 \\ +J_{wS}\left(x\sigma_S'^2S^2 + y\sigma_{SF}'SF - zBG\sigma_{Si}'\sqrt{i}S\right) \\ +J_{wi}\left(x\sigma_{Si}'\sqrt{i}S + y\sigma_{Fi}'\sqrt{i}F + zBGi\sigma_i^2\right) \\ +J_{Si}\sigma_{Si}'\sqrt{i}S \end{array}\right] \qquad (15)$$

where:

$\sigma_S'^2 = \sigma_S^2 + \lambda_S^2\delta_S^2 + 2\lambda_S\sigma_{S,qS}$ is the unconditional (on the absence of jumps) variance of the spot return;

$\sigma_F'^2 = \sigma_F^2 + \lambda_F^2\delta_F^2 + 2\lambda_F\sigma_{F,qF}$ is the unconditional (on the absence of jumps) variance of the futures return;

$\sigma'_{SF} = \sigma_{SF} + \lambda_S\sigma_{F,qS} + \lambda_F\sigma_{S,qF} + \lambda_S\lambda_F\sigma_{qS,qF}$ is the unconditional (on the absence of jumps) covariance between the spot and the futures returns;

$\sigma'_{Si} = \sigma_{Si} + \lambda_S\sigma_{i,qS}$ is the unconditional (on the absence of jumps) covariance between the spot return and the interest rate;

$\sigma'_{Fi} = \sigma_{Fi} + \lambda_F\sigma_{i,qF}$ is the unconditional (on the absence of jumps) covariance between the futures return and the interest rate;

$\sigma_S^2$ is the variance of the jumps in spot price, measured as returns on the spot price; and

$\delta_F^2$ is the variance of the jumps in futures price, measured as returns on the futures price;

The optimality conditions are obtained by taking the partial derivatives of (15) with respect to $C$, $x$, $y$, and $z$:

$$U_c - J_w = 0, \qquad (16)$$

$$\begin{aligned} & J_w(\alpha_S - \lambda_S k_S)S \\ +J_{ww}&\left(x\sigma_S'^2 S^2 + y\sigma_{SF}' SF - z\sigma_{Si}' BG\sqrt{i}\,S\right) = 0, \\ & +J_{wS}\sigma_S'^2 S^2 + J_{wi}\sqrt{i}\sigma_{Si}' S \end{aligned} \tag{17}$$

$$\begin{aligned} & J_w(\alpha_F - \lambda_F k_F)F \\ +J_{ww}&\left(y\sigma_F'^2 F^2 + x\sigma_{SF}' SF - z\sigma_{Fi}' BG\sqrt{i}\,F\right) = 0, \\ & +J_{wS}\sigma_{SF}' SF + J_{wi}\sqrt{i}\,\sigma_{Fi}' F \end{aligned} \tag{18}$$

$$\begin{aligned} & J_w Bi(1-\pi G) \\ +J_{ww}&\left(z\sigma_i^2 B^2 G^2 i - x\sigma_{Si}' BG\sqrt{i}\,S - y\sigma_{Fi}' BG\sqrt{i}\,F\right) = 0. \\ & +J_{wS}\left(-BG\sqrt{i}\,\sigma_{Si}' S\right) + J_{wi} BGi\sigma_i^2 \end{aligned} \tag{19}$$

Equation (16) is the familiar optimal condition of equality between marginal utility of consumption and marginal utility of wealth or more properly value function in our case. Dividing equations (17), (18) and (19) respectively by $S$, $F$ and $B$ clearly reveals that they are keys to give us the optimal solutions for, respectively $x^*$, the optimal number of spot contracts, $y^*$ the optimal number of futures contract, and $z^*$ the optimal number of default-free bonds. Solving simultaneously (17), (18) and (19) for $x^*$, $y^*$ and $z^*$, one obtains:

$$\begin{aligned} x^* = &-(A\Omega S)^{-1}\left[E_S R_{Fi} + E_F R_i + E_B G^{-1} R_F\right] - H_S G^{-1} \\ &-2H_i(\Omega S)^{-1}\sqrt{i}\,\sigma_i^2 R_F \end{aligned} \tag{20}$$

$$\begin{aligned} y^* = &-(A\Omega F)^{-1}\left[E_F R_{Si} + E_S R_i + E_B G^{-1} R_S\right] \\ &-2H_i(\Omega F)^{-1}\sqrt{i}\,\sigma_i^2 R_S \end{aligned} \tag{21}$$

$$\begin{aligned} z^* = &-(A\Omega BG)^{-1}\left[\sqrt{i}^{-1} E_S R_F + \sqrt{i}^{-1} E_F R_S + E_B G^{-1} R_{SF}\right] \\ &-H_i(\Omega BG)^{-1}\overline{\Omega} \end{aligned} \tag{22}$$

where:

$$A = -\frac{J_{WW}}{J_W}$$

$$H_S = -\frac{J_{WS}}{J_{WW}}$$

$$H_i = -\frac{J_{Wi}}{J_{WW}}$$

$$E_S = \alpha_S - \lambda_S k_S$$

$$E_F = \alpha_F - \lambda_F k_F$$

$$E_B = \sqrt{i}\,(1 - \pi G)$$

$$R_S = \sigma_S'^2 \sigma_{Fi}' - \sigma_{Si}' \sigma_{SF}'$$

$$R_F = \sigma_F'^2 \sigma_{Si}' - \sigma_{Fi}' \sigma_{SF}'$$

$$R_i = \sigma_{Si}' \sigma_{Fi}' - \sigma_i'^2 \sigma_{SF}'$$

$$R_{Si} = \sigma_S'^2 \sigma_i'^2 - \sigma_{Si}'^2$$

$$R_{Fi} = \sigma_F'^2 \sigma_i'^2 - \sigma_{Fi}'^2$$

$$R_{SF} = \sigma_S'^2 \sigma_F'^2 - \sigma_{SF}'^2$$

$$\Omega = \sigma_{Fi}'^2 \sigma_S'^2 - \sigma_F'^2 \sigma_i'^2 \sigma_S'^2 + \sigma_i'^2 \sigma_{SF}'^2 - 2\sigma_{Fi}' \sigma_{SF}' \sigma_{Si}' + \sigma_F'^2 \sigma_{Si}'^2$$

$$\overline{\Omega} = \sigma_{Fi}'^2 \sigma_S'^2 + \sigma_F'^2 \sigma_i'^2 \sigma_S'^2 - \sigma_i'^2 \sigma_{SF}'^2 - 2\sigma_{Fi}' \sigma_{SF}' \sigma_{Si}' + \sigma_F'^2 \sigma_{Si}'^2$$

Note that $A$ is the traditional Arrow-Pratt measure of absolute risk aversion, but the utility function is replaced here by the value function, and that $E_S$, $E_F$, $E_B$ are unconditional expected values of the returns on the spot contract, the futures contract and the default-free bond, respectively. The ratios $H_S$ and $H_i$ have as numerators, respectively, the variations in the marginal value function with respect to wealth, $J_W$, relative to the spot price, i.e. $J_{WS}$, and to the interest rate, i.e. $J_{Wi}$. While their common denominator is the second derivative of the value function with respect to wealth, i.e. $J_{WW}$. The ratio $H_S$ ($H_i$) thus measures the increase (decrease) in the marginal value function relative to wealth. More generally, it measures the increase (decrease) in wealth due to an increase in the spot price (interest rate). To further simplify the expressions in (20), (21) and (22), let us define:

$$E_X = S^{-1}[E_S R_{Fi} + E_F R_i + E_B G^{-1} R_F],$$

$$E_Y = F^{-1}[E_F R_{Si} + E_S R_i + E_B G^{-1} R_S],$$

$$E_Z = (BG\sqrt{i})^{-1}\left[E_S R_F + E_F R_S + E_B G^{-1}\sqrt{i}\, R_{SF}\right],$$

$$V_X = 2S^{-1}\sqrt{i}\sigma_i^2 R_F,$$

$$V_Y = 2F^{-1}\sqrt{i}\sigma_i^2 R_S,$$

$$V_Z = (BG)^{-1}\overline{\Omega}.$$

Then we can rewrite (20), (21) and (22) as:

$$x^* = -\Omega^{-1}(A^{-1}E_X + H_S G^{-1}\Omega + H_i V_X) \qquad (23)$$

$$y^* = -\Omega^{-1}(A^{-1}E_Y + H_i V_Y) \qquad (24)$$

$$z^* = -\Omega^{-1}(A^{-1}E_Z + H_i V_Z) \qquad (25)$$

From these expressions, one can devise the following hedge ratios:

$$h_{4F} = \frac{y^*}{x^*} = \frac{A^{-1}E_Y + H_i V_Y}{A^{-1}E_X + H_S G^{-1}\Omega + H_i V_X} \qquad (26)$$

$$h_{4B} = \frac{z^*}{x^*} = \frac{A^{-1}E_Z + H_i V_Z}{A^{-1}E_X + H_S G^{-1}\Omega + H_i V_X} \qquad (27)$$

where $h_{4F}$ and $h_{4B}$ are respectively the number of futures contracts and the number of default-free bonds to cover one spot contract to hedge against spot price risk, basis risk and settlement risk. These two hedge ratios should be compared with those proposed by Chang, Chang and Fang (1996b):[11]

$$h_{3F} = \frac{-\sigma_S J_w R_2}{\sigma_F\left(J_w R_1 + J_{wS}\sigma_S^2 R\right)} = \frac{-\sigma_S}{\sigma_F}\frac{A^{-1}R_2}{A^{-1}R_1 + H_S\sigma_S^2 R} \qquad (28)$$

$$h_{3B} = \frac{-\sigma_S(J_w R_3 + J_{wi}\sqrt{i}\,\sigma_i\,\sigma_S R)}{BG\sqrt{i}\,\sigma_i\left(J_w R_1 + J_{wS}\sigma_S^2 R\right)} \qquad (29)$$

$$= \frac{-1}{B}\frac{\sigma_S}{G\sqrt{i}\,\sigma_i}\frac{A^{-1}R_3 + H_i\sqrt{i}\,\sigma_i\,\sigma_S R}{A^{-1}R_1 + H_S\sigma_S^2 R}$$

where:

$$R = 1 - r_1^2 - r_2^2 - r_3^2 + 2\,r_1 r_2 r_3$$

$$R_1 = \alpha_s\left[\left(1 - r_2^2\right) - \theta_1\left(r_1 - r_2 r_3\right) + \theta_2\left(r_1 r_2 - r_3\right)\right]$$

$$R_2 = \alpha_s\left[\left(r_1 - r_2 r_3\right) - \theta_1\left(1 - r_3^2\right) + \theta_2\left(r_2 - r_1 r_3\right)\right]$$

$$R_3 = \alpha_s\left[\left(r_1 r_2 - r_3\right) - \theta_1\left(r_2 - r_1 r_3\right) + \theta_2\left(1 - r_1^2\right)\right]$$

$$\theta_1 = \frac{\alpha_F\,\sigma_S}{\sigma_F\,\alpha_S}$$

$$\theta_2 = \frac{-\left[\sqrt{i}\,(1 - \pi G)G^{-1}\right]\sigma_S}{\sigma_i\,\alpha_S}$$

$r_1$ is the correlation coefficient between the spot return and the futures return;

$r_2$ is the correlation coefficient between the futures return and the interest rate, $i$; and

$r_3$ is the correlation coefficient between the spot return and the interest rate, $i$.

Using a similar notation, equations (26) and (27) can be expanded into:[12]

$$h_{4F} = -\frac{S}{F}\,\frac{\sigma_S'}{\sigma_F'}\,\frac{A^{-1}R'_2 + 2\,H_i\sqrt{i}\,\sigma_i\left(r_2 - r_1 r_3\right)}{A^{-1}R'_1 - H_S G^{-1}\dfrac{\sigma_S'}{S}R + 2\,H_i\sqrt{i}\,\sigma_i\left(r_3 - r_1 r_2\right)} \qquad (30)$$

$$h_{4B} = -\frac{S}{B}\,\frac{\sigma_S'}{G\sqrt{i}\sigma_i}\,\frac{(A^{-1}R_3' - H_i\sqrt{i}\,\sigma_i R')}{A^{-1}R_1' - H_S G^{-1}\dfrac{\sigma_S'}{S}R + 2\,H_i\sqrt{i}\,\sigma_i\left(r_3 - r_1 r_2\right)} \qquad (31)$$

where:

$$R' = 1 - r_1'^2 + r_2'^2 + r_3'^2 - 2\,r_1' r_2' r_3'$$

$$R_1' = \frac{\sigma_S'}{E_S}\left[\left(1 - r_2'^2\right) - \theta_1'\left(r_1' - r_2' r_3'\right) + \theta_2'\left(r_1' r_2' - r_3'\right)\right]$$

$$R_2' = \frac{\sigma_S'}{E_S}\left[\left(r_1' - r_2' r_3'\right) - \theta_1'\left(1 - r_3'^2\right) + \theta_2'\left(r_2' - r_1' r_3'\right)\right]$$

$$R_3' = \frac{\sigma_S'}{E_S}\left[\left(r_1'r_2' - r_3'\right) - \theta_1'\left(r_2' - r_1'r_3'\right) + \theta_2'\sqrt{i}\left(1 - r_1^2\right)\right]$$

$$\theta_1' = \frac{E_F\,\sigma_S'}{\sigma_F'\,E_S}$$

$$\theta_2' = \frac{-E_B G^{-1}\,\sigma_S'}{\sigma_i\,E_S} = \frac{-\left[\sqrt{i}\,(1-\pi G)G^{-1}\right]\sigma_S'}{\sigma_i\,E_S}$$

$r_1'$ is the correlation coefficient between the spot return and the futures return, conditional on the absence of jumps in both the spot and the futures prices;

$r_2'$ is the correlation coefficient between the futures return and the interest rate $i$, conditional on the absence of jumps in the futures price; and

$r_3'$ is the correlation coefficient between the spot return and the interest rate $i$, conditional on the absence of jumps in the spot price.

Because in our model a jump process is added to both the spot return and futures returns processes, the instantaneous means ($E_S$, $E_F$), the instantaneous standard deviations of these two processes ($\sigma_S'$, $\sigma_F'$) as well as the covariances between the processes ($\sigma_{SF}'$), and between the processes and the interest rate ($\sigma_{Si}'$, $\sigma_{Fi}'$) are now different from the Chang, Chang and Fang (1996b) model. The same could be said of the return per unit of risk on the futures contract relative to the return per unit of risk on the spot contract ($\theta_1'$), and the return per unit of risk on the bond relative to the return per unit of risk on the spot contract ($\theta_2'$). However, beyond these minor and obvious dissimilarities, the structure of equations (30) and (31) is very different from equations (28) and (29). First, a term involving $H_i$ appears in both the numerators and denominators of both $h_{4F}$ and $h_{4B}$, while it appears only in the numerator of $h_{3B}$, with an opposite sign. Second, the collections of correlation coefficients designated $R$, $R_1$, $R_2$, and $R_3$ in $h_{3F}$ and $h_{3B}$ look much like $R'$, $R_1'$, $R_2'$, and $R_3'$ in $h_{4F}$ and $h_{4B}$, but in fact are quite different, because the latter are multiplied by $\sigma_S'/E_S$, the reciprocal of the reward to risk ratio of the spot contract (unconditional on the absence of jumps),[13] and not by $\alpha_S$, a measure of performance of the spot contract, as in the former measures. Furthermore, while $R$ is also used in the denominator of $h_{4F}$ and $h_{4B}$, it is also multiplied by $\sigma_S'/E_S$ and not by $\sigma_S^2$ as in $h_{3F}$ and $h_{3B}$. Lastly, the ratio of the spot price $S$ to the futures price $F$ is a determinant in $h_{4F}$, and the ratio of the spot price $S$ to the default-free bond price $B$ is a determinant in $h_{4B}$. In contrast, only the bond price $B$ appears in the denominator of $h_{3B}$. Therefore, provided that $A^{-1}$, $H_S$ and $H_i$ are constant, it is clear that $h_{4F}$ and $h_{4B}$ are functions of $S$, $F$, $i$ (or $B$) and $t$, while $h_{3F}$ is a constant, and $h_{3B}$ is a function of only $i$ (or $B$) and $t$.[14] These differences mainly result from our assumptions of geometric processes followed by $S$ and $F$, while the assumption of a Cox, Ingersoll and Ross' (1985) model for $i$ also induces a geometric process for $B$. While Chang, Chang and Fang (1996b) assumed arithmetic processes for $S$ and $F$, they adopted Cox, Ingersoll and Ross' model for $i$, and thus obtained a term involving $B$ in $h_{3B}$ in equation (29). Of course, if one assumes that the cost-of-carry (default-free interest rate plus storage cost minus income rate) and the convenience yield are nil, then $S/F = 1$, and that ratio will disappear from (30). Nonetheless, it is difficult to see how the ratio $S/B$ could become $1$ in (31).

To specialize the hedge ratios $h_{4F}$ and $h_{4B}$, let us assume, as did Chang, Chang and Fang (1996b), that the utility function is logarithmic. This assumption implies that $J_{wS} = J_{wi} = 0$, and the hedge ratios become:

$$h_{4F}^{(a)} = -\frac{S}{F}\frac{\sigma_S'}{\sigma_F'}\frac{R'_2}{R'_1} \tag{32a}$$

$$h_{4B}^{(a)} = -\frac{S}{B}\frac{\sigma_S'}{G\sqrt{i}\,\sigma_i}\frac{R_3'}{R_1'}. \tag{33}$$

By comparison, Chang, Chang and Fang (1996b) has an equivalent to equation (32a):[15]

$$h_{3F}^{(a)} = -\frac{\sigma_S}{\sigma_F}\frac{R_2}{R_1}. \tag{32b}$$

Equations (32a) and (32b) are almost identical, except for the presence in (32a) of the ratio $S/F$ due to our assumption that the spot price and the futures price follow geometric generalized Brownian motions, and for the obvious dissimilarities in the instantaneous means and standard deviations of these processes, because of our assumption that such processes have additional jump risks. Yet if one assumes away the jump risks, i.e. let $\lambda_S = \lambda_F = 0$, then our hedge ratio still differs from (32b) by the ratio $S/F$:

$$h_{4F}^{(aa)} = \frac{S}{F}h_{3F}^{(a)}.$$

However, empirical evidence overwhelmingly points to jump risks in almost all financial assets, and therefore it is unlikely that, empirically $\lambda_S$ and $\lambda_F$ will simply collapse to zero. In that case, even when for a given period of time there is no jump, however defined, in the spot or in the futures price, the values of the distribution moments conditional on the absence of jumps are still different from the unconditional, or inclusive of all observations, values. Hence there is another fundamental difference between the two hedge ratios $h_{4F}^{(aa)}$ and $h_{3F}^{(a)}$. The first two moments of the spot and futures return processes, respectively $\alpha_S$ and $\alpha_F$, and $\sigma_S$ and $\sigma_F$, are not the same in the Chang, Chang and Fang (1996a) model as in ours. Similarly, the three correlation coefficients $r_1$, $r_2$, and $r_3$ in Chang, Chang and Fang (1996a) are not the same as $r'_1$, $r'_2$, and $r'_3$ in our model. In our hedge ratio, the processes' first two moments along with the correlation coefficients are conditional on the absence of jumps in both the spot and futures prices. In the Chang, Chang and Fang (1996b) hedge ratio, the correlation coefficients are inclusive of all observations. [16] More specifically, one would obtain:

$$h_{4F}^{(aa)} = -\frac{S}{F}\frac{\sigma_S}{\sigma_F}\frac{(r'_1 - r'_2 r'_3) - \theta_1(1 - r_3'^2) + \theta_2(r'_2 - r'_1 r'_3)}{(1 - r_2'^2) - \theta_1(r'_1 - r'_2 r'_3) + \theta_2(r'_1 r'_2 - r'_3)} \neq h_{3F}^{(a)} \quad (32c)$$

where all distribution moments are conditional on the absence of jumps in both the spot and the futures prices. By invoking unbiasedness of the futures price relative to the spot price, one has $E_F = \theta'_1 = 0$ and equations (32a) and (33) become, respectively:

$$h_{4F}^{(b)} = -\frac{S}{F}\frac{\sigma'_S}{\sigma'_F}\frac{(r_1 - r_2 r_3) + \theta'_2(r_2 - r_1 r_3)}{(1 - r_2^2) + \theta'_2(r_1 r_2 - r_3)}, \quad (34a)$$

$$h_{4B}^{(b)} = -\frac{S}{B}\frac{\sigma'_S}{G\sqrt{i}\,\sigma_i}\frac{(r_1 r_2 - r_3) + \theta'_2\sqrt{i}(1 - r_1^2)}{(1 - r_2^2) + \theta'_2(r_1 r_2 - r_3)}. \quad (35)$$

Chang, Chang and Fang (1996b) have an equivalent to equation (34a):[17]

$$h_{3F}^{(b)} = -\frac{\sigma_S}{\sigma_F}\frac{(r_1 - r_2 r_3) + \theta_2(r_2 - r_1 r_3)}{(1 - r_2^2) + \theta_2(r_1 r_2 - r_3)}. \quad (34b)$$

The same comments made for (32a) versus (32b) can be repeated for (34a) versus (34b). If one assumes that the correlation coefficients between the spot return and the interest rate, and between the futures return and the interest rate are nil, i.e. $r_2 = r_3 = 0$, then equations (32a) and (33) become:

$$h_{4F}^{(c)} = -\frac{S}{F}\frac{\sigma'_S}{\sigma'_F}\frac{r_1 - \theta'_1}{1-\theta'_1 r_1} \tag{36a}$$

$$h_{4B}^{(c)} = -\frac{S}{B}\frac{\sigma'_S}{G\sqrt{i}\,\sigma_i}\frac{\theta'_2\left(1-r_1^2\right)\sqrt{i}}{1-\theta'_1 r_1}. \tag{37}$$

The equivalent to equation (36a) in Chang, Chang and Fang (1996b) is:[18]

$$h_{3F}^{(c)} = -\frac{\sigma_S}{\sigma_F}\frac{r_1 - \theta_1}{1-\theta_1 r_1}. \tag{36b}$$

Moreover, by again invoking the unbiased nature of the futures price relative to the spot price, i.e. $E_F = \theta'_1 = 0$, equations (36a) and (37) become:

$$h_{4F}^{(d)} = -\frac{S}{F}\frac{\sigma'_S}{\sigma'_F}\, r_1 \tag{38a}$$

$$h_{4B}^{(d)} = -\frac{S}{B}\frac{\sigma'_S}{G\sqrt{i}\,\sigma_i}\,\theta'_2\left(1-r_1^2\right)\sqrt{i}. \tag{39}$$

In Chang, Chang and Fang (1996b), the equivalent to equation (38a) is:[19]

$$h_{3F}^{(d)} = -\frac{\sigma_S}{\sigma_F}\, r_1. \tag{38b}$$

Equation (38b) is the traditional regression hedge ratio, obtained as one, seeks to minimize the variance of the return on the hedged portfolio containing one spot contract and the optimal number of futures contracts. By contrast, equation (38a) implies that the cost-of-carry and convenience yield, represented by the basis *S/F*, must also be taken into account in deriving the hedge ratio, even in that basic framework.

It should be noted that in all its four variations, the optimal number of default-free bonds to hold for hedging against basis and settlement risks in (33), (35), (37) and (39), is not nil even when the utility function is logarithmic, and when there is no correlation between the interest rate and the spot return or the futures return. Ultimately, that optimal number of default-free bonds depends on the spot price relative to the bond price, on the ratio of the unconditional standard deviation of the spot price $\sigma'_S$ to the volatility term of the bond price $G\sqrt{i}\ \sigma_i$, on the reward-to-risk ratio of the bond relative to that of the spot contract, on the complement to the determination coefficient between the spot and the futures prices, and lastly on the square root of the interest rate, as (39) can be rewritten as:

$$h_{4B}^{(d)} = \frac{S}{B}\,\frac{\sigma_S'^2}{G^2\, i\,\sigma_i^2}\,\frac{i(1-\pi G)}{E_S}\left(1-r_1^2\right)\sqrt{i}. \tag{40}$$

## A NUMERICAL EXAMPLE

To illustrate the theoretical results obtained so far, we have used price data on the West Texas Intermediate (WTI) crude oil spot contract for $S$, price data on the front month futures contract on WTI traded on the New York Mercantile Exchange (NYMEX) for $F$ and yields on 3-month U.S. Treasury bill for $i$.[20] The data covered the period of November 4, 1993 to April 20, 1998 inclusive, for 1113 daily observations. The sample has been divided into two sub-periods: The first one ranges from November 4, 1993 to May 2, 1997 inclusive (873 observations) and is used to estimate the parameters of the stochastic processes for $S$, $F$ and $i$; the second one, which comprises the remaining observations, is used to compute the hedge ratios.

The spot and futures returns are computed as first differences in the logarithms of the respective price series. A jump in the spot and futures returns is defined as any observation which is more than one standard deviation in absolute value from the mean, computed by using the whole return series. The means and standard deviations of the spot and futures returns, computed by excluding the jumps, are respectively the means and standard deviations conditional on the absence of jumps, $\alpha_S$, $\alpha_F$, $\sigma_S$ and $s_F$. Using the series of spot and futures returns containing only the observations deemed to be jumps, the probabilities of a jump for the spot and futures prices, respectively $\lambda_S$ and $\lambda_F$, are computed as the average numbers of jumps per daily unit of time. Therefore, $\lambda_S$ and $\lambda_F$ are computed as the number of days during the sample period where a jump is observed divided by the sample size (in days). The means and standard deviations of the jumps in the spot and futures returns, respectively $\mu_S$, $\mu_F$, $\delta_S$ and $\delta_F$ are computed from the series of spot and futures returns containing only the observations deemed to be jumps. A successive series of units of time constitutes a distribution of jumps which tend increasingly towards a lognormal distribution with the time length of the series. Therefore, let $X_S$ be a random variable representing the size of a jump in the spot price, and since we have:

$$Log(X_S) \sim N(\mu_S, \delta_S^2),$$

Then, the expected size of a jump, $k_S$ is :

$$k_S = E(X_S) = Exp\left[\mu_S - \frac{\delta_S^2}{2}\right] - 1.$$

The same applies to $k_F$. Table 1 presents the results of our computations for the parameters of the spot and futures prices.

**TABLE 1**
**PARAMETERS OF THE SPOT AND FUTURES PRICE PROCESSES**

| $\alpha_S$ | $\alpha_F$ | $\lambda_S$ | $\lambda_F$ | $k_S$ | $K_F$ | $E_S$ | $E_F$ |
|---|---|---|---|---|---|---|---|
| 4.64% | 4.91% | 25.23% | 24.42% | -14.26% | -13.15% | 8.23% | 8.12% |

$\alpha_S$ is the instantaneous expected spot return;

$\alpha_F$ is the instantaneous expected futures return;

$\lambda_S$ is the expected number of jumps (or probability of a jump) in the spot price at each point in time;

$\lambda_F$ is the expected number of jumps (or probability of a jump) in the futures price at each point in time;

$k_S$ is the expected size of the jumps in spot price, measured by the change in the spot price;

$k_F$ is the expected size of the jumps in futures price, measured by the change in the futures price;

$E_S = \alpha_S - \lambda_S k_S$; and

$E_F = \alpha_F - \lambda_F k_F$

To compute the parameters of the mean reverting interest rate process, we adopt the following discrete model, which is compatible with the process specified in (5):

$$i_t = k\alpha_i + (1-k)i_{t-1} + \varepsilon_t.$$

The discrete model has been estimated through an OLS regression, and the estimated slope is .818, with a t-value of 12.55, implying that the mean reverting speed $k$ is .182. The estimated constant is .832, with a t-value of 3.00, which implies that the long term instantaneous mean of the interest rate $\alpha_i$ is 4.57%. The standard deviation of the interest rate process is $\sigma(\varepsilon_t)$ = .04%, and the coefficient of determination is .961. The market risk parameter of interest rates, $\pi$, is computed as the slope in the regression of the bond price $B(t, T)$ over the basis $Ln[S(t)/F(t)]$; as expected, it is generally close to zero and not significant whatever the period and the length of the period over which the regression is estimated.

Table 2 presents the conditional (on the absence of jumps) variance-covariance matrix of the spot return, the futures return and the interest rate, while Table 3 presents its unconditional equivalent. Table 4 presents the same variance-covariance matrix when no distinction is made between jumps and no-jumps among the spot and futures return observations. The values in that matrix are input to the computation of hedge ratios according to Chang, Chang and Fang (1996b), and should be compared with the unconditional values of Table 3. It appears that to distinguish between jumps and no-jumps in the spot and the futures return observations, using the definition of a jump as a departure of more than one standard deviation from the mean results in underestimation of the variances and covariances. This is because the values in Table 3 are about half of their counterparts in Table 4, except for the covariance between the futures return and the interest rate, which is about one tenth of its counterpart from Table 4.

**TABLE 2**

**CONDITIONAL (ON THE ABSENCE OF JUMPS) VARIANCE-COVARIANCE MATRIX OF THE SPOT RETURN, THE FUTURES RETURN AND THE INTEREST RATE**

| | | | | |
|---|---|---|---|---|
| $\sigma_S^2 = 6.76\ 10^{-2}$ | $\sigma_{SF} = 5.76\ 10^{-2}$ | $\sigma_{S,qS} = 2.45\ 10^{-2}$ | $\sigma_{S,qF} = 2.08\ 10^{-2}$ | $\sigma_{Si} = 4.81\ 10^{-7}$ |
| - | $\sigma_F^2 = 6.05\ 10^{-2}$ | $\sigma_{F,qS} = 1.97\ 10^{-2}$ | $\sigma_{F,qF} = 2.27\ 10^{-2}$ | $\sigma_{Fi} = -9.04\ 10^{-8}$ |
| - | - | $\delta_S^2 = 3.40\ 10^{-2}$ | $\sigma_{qS,qF} = 2.23\ 10^{-2}$ | $\sigma_{i,qS} = -1.74\ 10^{-8}$ |
| - | - | - | $\delta_F^2 = 3.07\ 10^{-2}$ | $\sigma_{i,qF} = 5.81\ 10^{-7}$ |
| - | - | - | - | $\sigma_i^2 = -1.66\ 10^{-7}$ |

$\sigma_S^2$ is the variance of the spot return, conditional on the absence of jumps in the spot price;

$\sigma_F^2$ is the variance of the futures return, conditional on the absence of jumps in the futures price;

$\delta_S^2$ is the variance of the jumps in spot price, measured as returns on the spot price;

$\delta_F^2$ is the variance of the jumps in futures price, measured as returns on the futures price;

$\sigma_i^2$ is the variance of the interest rate;

$\sigma_{SF}$ is the covariance between the spot and futures returns, conditional on the absence of jumps in both the spot and the futures prices;

$\sigma_{S,qS}$ is the covariance between the spot return, conditional on the absence of jumps in the spot price, and the jump in the spot price;

$\sigma_{S,qF}$ is the covariance between the spot return, conditional on the absence of jumps in the spot price, and the jump in the futures price;

$\sigma_{Si}$ is the covariance between the spot return, conditional on the absence of jumps in the spot price, and the interest rate;

$\sigma_{F,qS}$ is the covariance between the futures return, conditional on the absence of jumps in the futures price, and the jump in the spot price;

$\sigma_{F,qF}$ is the covariance between the futures return, conditional on the absence of jumps in the futures price, and the jump in the futures price;

$\sigma_{Fi}$ is the covariance between the futures return, conditional on the absence of jumps in the futures price, and the interest rate;

$\sigma_{qS,qF}$ is the covariance between the jump in the spot price and the jump in the futures price;

$\sigma_{i,qS}$ is the covariance between the interest rate and the jump in the spot price; and

$\sigma_{i,qF}$ is the covariance between the interest rate and the jump in the futures price.

**TABLE 3**
**UNCONDITIONAL (ON THE ABSENCE OF JUMPS) VARIANCE-COVARIANCE MATRIX OF THE SPOT RETURN, THE FUTURES RETURN AND THE INTEREST RATE**

| | | |
|---|---|---|
| $\sigma'^2_S = 8.21\ 10^{-2}$ | $\sigma'_{SF} = 6.90\ 10^{-2}$ | $\sigma'_{Fi} = 4.77\ 10^{-7}$ |
| - | $\sigma'^2_F = 7.35\ 10^{-2}$ | $\sigma'_{Si} = 5.15\ 10^{-8}$ |
| - | - | $\sigma^2_i = 1.66\ 10^{-7}$ |

$\sigma'^2_S = \sigma^2_S + \lambda^2_S\delta^2_S + 2\lambda_S\sigma_{Sqs}$ is the unconditional (on the absence of jumps) variance of the spot return;

$\sigma'^2_F = \sigma^2_F + \lambda^2_F\delta^2_F + 2\lambda_F\sigma_{F\,qs}$ is the unconditional (on the absence of jumps) variance of the futures return;

$\sigma'_{SF} = \sigma_{SF} + \lambda_S\sigma_{F,qs} + \lambda_F\sigma_{S,qs} + \lambda_S\lambda_F\sigma_{qS,qF}$ is the unconditional (on the absence of jumps) covariance between the spot and the futures returns;

$\sigma'_{Si} = \sigma_{Si} + \lambda_S\sigma_{i,qs}$ is the unconditional (on the absence of jumps) covariance between the spot return and the interest rate;

$\sigma'_{Fi} = \sigma_{Fi} + \lambda_F\sigma_{i,qF}$ is the unconditional (on the absence of jumps) covariance between the futures return and the interest rate;

**TABLE 4**
**VARIANCE-COVARIANCE MATRIX OF THE SPOT RETURN, THE FUTURES RETURN AND THE INTEREST RATE, WITH NO DISTINCTION BETWEEN JUMPS AND NO-JUMPS AMONG OBSERVATIONS**

| | ***dS/S*** | ***dF/F*** | ***i*** |
|---|---|---|---|
| ***dS/S*** | $15.07\ 10^{-2}$ | $12.05\ 10^{-2}$ | $4.64\ 10^{-7}$ |
| ***dF/F*** | – | $13.68\ 10^{-2}$ | $4.91\ 10^{-7}$ |
| ***i*** | – | – | $1.66\ 10^{-7}$ |

*dS/S* is the spot return;

*dF/F* is the futures return; and

*i* is the interest rate.

To further investigate the comparison between the unconditional values from our assumed jump-diffusion processes and the historical data, Table 5 presents the correlation coefficients for the three possible ways to look at the data: (1) observations exclusive of jumps, i.e. conditional on the absence of jumps; (2) observations unconditional on the absence of jumps, i.e. inclusive of jumps through their first two moments; and (3) all observations without distinction between jumps and no-jumps. While the correlation between the interest rate and either the spot return or the futures change is negligible, the correlation between the spot return and the futures return decreases with the degree of inclusion of the jumps in the sample. When the observations are exclusive of jump, that correlation is highest at 0.90, and when no distinction is made between jumps and no-jumps, it is lowest at 0.84. The econometric explanation is the effect of exclusion of jumps as outliers, while the economic explanation is that basis risk is probably higher when jumps are allowed to occur. The decrease in correlation illustrates the importance of modeling jumps in the diffusion processes of spot and futures returns. Moreover, because the occurrence of jumps usually coincides with period of liquidity gap, it is important to recognize that the interest rate is stochastic, and that settlement risk must be covered through bond holdings.

**TABLE 5**
**CORRELATION MATRIX OF THE SPOT RETURN, THE FUTURES RETURN AND THE INTEREST RATE**

| | *dS/S* | *dF/F* | *i* |
|---|---|---|---|
| *dS/S* | 1 | (1) 0.90* | (1) 4.54 $10^{-3}$ |
| | | (2) 0.89* | (2) 4.08 $10^{-3}$ |
| | | (3) 0.84* | (3) $2.90^{-3}$ |
| *dF/F* | _ | 1 | (1) -0.90 $10^{-3}$ |
| | | | (2) 0.46 $10^{-3}$ |
| | | | (3) 3.20 $10^{-3}$ |
| *i* | _ | _ | 1 |

*dS/S* is the spot return;

*dF/F* is the futures return;

*i* is the interest rate;

(1) refers to the values conditional on the absence of jumps, as presented in Table 2;

(2) refers to the values unconditional on the absence of jumps, as presented in Table 3; and

(3) refers to the values with no distinction between jumps and no-jumps among observations, as presented in Table 4.

* Significant at the .05 level.

The hedge ratios have been computed for different hedging horizons, using May 5, 1997 as the date for the futures position initialization (t = 0), and the results are presented in Table 6.[21] Hedge ratios derived by assuming unbiasedness, i.e. $E_F = \alpha_F = 0$, have not been computed, since in our case this assumption can not be supported, as $\alpha_F = 4.91\%$ and $E_F = 8.12\%$. Also, hedge ratios derived by assuming that $r_2 = r_3 = 0$ have not been presented, since they differ little from $h_{4F}^{(a)}$, because of the very small values of $r_2$ and $r_3$ (see Table 5). Hedge ratio $h_1$ yields a value of 105.85%, which is very inefficient relative to the hedge ratios reported in Table 6. Finally, hedge ratios $h_2$ and $h_{3F}^{(d)}$ cannot be computed for varying hedging horizons, since they are derived from essentially one-period model.

Table 6 reveals that the values for our hedge ratios, $h_{4F}^{(a)}$ and $h_{4F}^{(aa)}$, as well as for the traditional regression hedge ratio, $h_{3F}^{(d)}$, are all negative and, in the case of the first two ratios, decrease in absolute values with the length of the hedging horizon. The decrease in absolute values with the length of the hedging horizon can be explained by the greater risk involved, and thus the lower efficiency of the optimal hedge ratio. Remarkably, the traditional regression hedge ratio for a hedging horizon of one day is close, yet lower in absolute values, to our hedge ratio $h_{4F}^{(a)}$ for hedging horizons of day up to one week. However, only for hedging horizons of two weeks or more does the traditional regression hedge ratio have higher absolute values than $h_{4F}^{(a)}$. The hedge ratio $h_{4F}^{(aa)}$ is strictly higher in absolute values than either $h_{4F}^{(a)}$ or $h_{3F}^{(d)}$ for all hedging horizons, and is close to –1. The reason for this is the exclusion in $h_{4F}^{(aa)}$ of jumps in the spot and the futures returns, which therefore implies that jump risk is ignored, and thus permits a higher level of efficiency in the hedge ratios. These observations are all reassuring since our hedge ratios are close to, and yet (generally for $h_{3F}^{(d)}$ and strictly for $h_{4F}^{(aa)}$) more efficient than the simple traditional regression hedge ratio. By contrast, the hedge ratios proposed by Chang, Chang and Fang (1996a, b), respectively $h_2$ and $h_{3F}^{(a)}$, are positive, and in the case of the latter, increasing in absolute values (i.e. increasing inefficiency) with the length of the hedging horizon. It should be noted however that in Chang, Chang and Fang (1996b), the empirical evidence, which points to relatively highly efficient hedge ratios, is derived from data on S&P 500 stock index, a financial asset which is not characterized by jump risk as much as the WTI crude oil contract which is used in this paper. Moreover, while Chang, Chang and Fang (1996a) takes into account jump risk, given that the basis is endogenous to their model, the jump risk is the same for the spot price than for the futures price, which severely limits the efficiency of their hedge ratio. In addition, because the Chang, Chang and Fang (1996b) model assumes, without taking into account jump risk, arithmetic generalized Brownian motion for both the spot and futures returns, it is not clear whether such diffusion models are better adapted to prices than to returns, or to neither.[22] The results so far seem to show that the Chang, Chang and Fang (1996b) hedge ratio is probably not well suited for highly volatile assets, or ones that entail a jump risk.

**TABLE 6**
**HEDGE RATIOS**

| Hedging horizons | $h_{4F}^{(a)}$ | $h_{4F}^{(aa)}$ | $h_{3F}^{(a)}$ | $h_1$ | $h_{3F}^{(d)}$ |
|---|---|---|---|---|---|
| **One day** | -89.51% | -96.87% | 26.83% | 26.75% | -88.04% |
| **Two days** | -89.38% | -96.81% | 27.13% | – | – |
| **Three days** | -89.25% | -96.76% | 27.43% | – | – |
| **One week** | -88.68% | -96.52% | 28.73% | – | – |
| **Two weeks** | -87.77% | -96.14% | 30.77% | – | – |
| **One month** | -86.14% | -95.46% | 34.44% | – | – |
| **One year** | -74.25% | -90.20% | 60.34% | – | – |

$h_{4F}^{(a)} = -\frac{S}{F}\frac{\sigma'_S}{\sigma'_F}\frac{R'_2}{R'_1}$ is our equation (32a), when the utility function is logarithmic; computational input is from Table 3;

$$h_{4F}^{(aa)} = -\frac{S}{F}\frac{\sigma_S}{\sigma_F}\frac{(r'_1 - r'_2 r'_3) - \theta_1(1 - r'^2_3) + \theta_2(r'_2 - r'_1 r'_3)}{(1 - r'^2_2) - \theta_1(r'_1 - r'_2 r'_3) + \theta_2(r'_1 r'_2 - r'_3)}$$ is our equation (32c), when the utility function is logarithmic and there is no jump in the spot and in the futures prices, i.e. $\lambda_S = \lambda_F = 0$; computational input is from Table 2;

$h_{3F}^{(a)} = -\dfrac{\sigma_S}{\sigma_F}\dfrac{R_2}{R_1}$ is our equation (32b) and the hedge ratio obtained by Chang, Chang and Fang (1996b, equation 16); computational input is from Table 4;

$h_1 = \dfrac{\left(\sigma_{SF} + j\sigma_{xy}\right)}{\left(\sigma_F^2 + j\sigma_y^2\right)}$ is our equation (1) and the hedge ratio obtained by Chang, Chang and Fang (1996a, equation 5); computational input is from Table 4; and

$h_{3F}^{(d)} = -\dfrac{\sigma_S}{\sigma_F} r_1$ is our equation (38b) and the traditional regression hedge ratio; computational input is from Table 4.

Hedging horizon is the horizon over which the investor intends to cover his or her cash position against the asset price risk. It is assumed that the hedging horizon coincides with the time to maturity of the futures contract.

In Table 7, efficiency measures of the hedge ratios are presented. The benchmark hedge ratios are those proposed in this paper, $h_{4F}^{(a)}$ (panel A of Table 7) and (panel B of Table 7). The efficiency measure is simply computed as:

$$\frac{BHR - EHR}{|BHR|}$$

where *BHR* is the benchmark hedge ratio and *EHR* is the evaluated hedge ratio. From Table 7, it is clear that both our hedge ratios are strictly more efficient than the hedge ratios proposed by Chang, Chang and Fang (1996a, b), and that efficiency is increasing with the length of the hedging horizon. Our hedging ratios are also more efficient than the traditional regression hedge ratio, although the efficiency is decreasing with the length of the hedging horizon. Moreover, for $h_{4F}^{(a)}$, i.e. for hedging horizons of two weeks or more, our hedge ratio is less efficient than the traditional regression hedge ratio. The reason for this is the short-term nature of our hedge ratios, developed in a continuous time framework.

## TABLE 7
## EFFICIENCY MEASURES OF HEDGE RATIOS

| Hedging horizons | $h_{3F}^{(a)}$ | $h_1$ | $h_{3F}^{(d)}$ |
|---|---|---|---|
| **Panel A Efficiency relative to $h_{4F}^{(a)}$** | | | |
| **One day** | -129.97% | -129.88% | -1.64% |
| **Two days** | -130.35% | -129.92% | -1.49% |
| **Three days** | -130.73% | -129.97% | -1.35% |
| **One week** | -132.39% | -130.16% | -0.71% |
| **Two weeks** | -135.05% | -130.47% | 0.31% |
| **One month** | -139.98% | -131.05% | 2.20% |
| **One year** | -181.27% | -136.03% | 18.59% |
| **Panel B Efficiency relative to $h_{4F}^{(aa)}$** | | | |
| **One day** | -7.60% | -127.69% | -9.11% |
| **Two days** | -7.67% | -128.02% | -9.05% |
| **Three days** | -7.76% | -128.34% | -9.01% |
| **One week** | -8.12% | -129.76% | -8.78% |
| **Two weeks** | -8.70% | -132.00% | -8.42% |
| **One month** | -9.76% | -136.07% | -7.77% |
| **One year** | -17.68% | -166.89% | -2.39% |

The efficiency measure is computed as: $\frac{BHR - EHR}{|BHR|}$.

Where *BHR* is the benchmark hedge ratio and *EHR* is the evaluated hedge ratio. For example, the efficiency measures in the first column of panel A are computed as:

$$\frac{h_{4F}^{(a)} - h_{3F}^{(a)}}{\left|h_{4F}^{(a)}\right|}.$$

Hedging horizon is the horizon over which the investor intends to cover his or her cash position against the asset price risk.

To further investigate the efficiency of the hedge ratio $h_{4F}^{(a)}$ over short horizons, we have computed it again for hedging horizons of one day to one week, and for an adjustment date of 7, 15, 30, and 180 days after the initial futures position initialization date of May 5, 1997. The purpose is to illustrate the dynamic nature of our hedge ratio, which implies continuous adjustment to the hedging position. The results are presented in Table 8. The first line in each cell is the value of the hedge ratio $h_{4F}^{(a)}$, and the second line is the efficiency measure of the traditional regression hedge ratio $h_{3F}^{(d)}$ relative to $h_{4F}^{(a)}$. The results clearly show that our hedge ratio is strictly more efficient than the traditional regression hedge ratio for short horizon and for adjustment delay as long as 180 days.

**TABLE 8**
**EFFICIENCY OF $h_{4F}^{(a)}$ OVER SHORT HORIZONS**

| | Futures position adjustment dates | | | |
|---|---|---|---|---|
| **Hedging horizons** | May 12, 1997 (7 days) | May 20, 1997 (15 days) | June 3, 1997 (30 days) | November 3, 1997 (180 days) |
| **One day** | (1) -89.63%<br>(2) -1.77% | (1) -89.14%<br>(2) -1.23% | (1) -89.29%<br>(2) -1.4% | (1) -90.60%<br>(2) -2.82% |
| **Two days** | (1) -89.50%<br>(2) -1.63% | (1) -89.01%<br>(2) -1.09% | (1) -89.16%<br>(2) -1.25% | (1) -90.47%<br>(2) -2.68% |
| **Three days** | (1) -89.37%<br>(2) -1.49% | (1) -88.88%<br>(2) -0.94% | (1) -89.03%<br>(2) -1.11% | (1) -90.34%<br>(2) -2.54% |
| **One week** | (1) -88.79%<br>(2) -0.84% | (1) -88.31%<br>(2) -0.30% | (1) -88.46%<br>(2) -0.47% | (1) -89.76%<br>(2) -1.91% |

The hedge position initialization date is May 5, 1997. Hedging horizon is the horizon over which the investor intends to cover his or her cash position against the asset price risk. It coincides with the time to maturity of the futures contract.

## CONCLUSION

In this paper, optimal hedge ratios are developed in a dynamic setting, where the spot and futures returns follow Poisson jump-diffusion processes, and where additional risks are created by the stochastic interest rate and the marking-to-market. We show that our hedge ratios are very different from those proposed by Chang,

Chang and Fang (1996b) and that they are truly dynamic in nature. Using data on the WTI crude oil contract, we also show that our hedge ratios are more efficient than the hedge ratios proposed by Chang, Chang and Fang (1996a, b) along with the traditional regression hedge ratio. While settlement risk is common to all assets, jump risk and basis risk are specific to each asset. Therefore, a natural extension of this paper is to assess the efficiency of the proposed hedge ratios examined in terms of a wide range of assets. It would be normal to expect that our hedge ratios are more appropriate for assets for which jump risk and basis risk are high.

The hedge ratios we derive are uniformly applicable to all firms with the utility attributes we describe in this paper. Yet when firms engage in hedging activities, they are predicated by other determinants, which explains the differentiation in hedging policies among firms of the same sector. Haushalter (1998) documents a wide variation in the hedging policies among 100 U.S. oil and gas producers from 1992 to 1994.[23] His evidence supports several reasons for corporate risk management: i) to alleviate financial contracting costs; ii) to benefit from economies of scale in hedging; and iii) to reduce the basis risk. Adam (1998) also supports the first rationale in the North American gold mining industry. He shows that the extent of hedging volume is related to the company capital expenditures, financial leverage and access to the financial markets. Another natural extension of this paper is to consider, along the lines of Haushalter and Adam, attributes other than wealth, and objectives other than wealth maximization, in the hedging program of the firms.

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## Notes

1. Raposo (1999) has recently provided a good review of the literature on corporate risk hedging. Johnson (1960) is probably the first scholarly treatment of the subject.

2. This is equation (5) in Chang, Chang and Fang (1996a), corrected for some typographical errors.

3. This is equation (14) in Chang, Chang and Fang (1996a).

4. Brennan (1991) explains basis risk by the stochastic nature of the cost-of-carry which is specific to each commodity or asset. Bailey and Chan (1993) use macroeconomic variables to explain basis risk.

5. Anderson & Danthine (1981) and Howard & D'Antonio (1984) treat futures hedging in a (one-period) risk-return framework.

6. Cox, Ingersoll & Ross (1981), Jarrow & Oldfield (1981), and Richard & Sundaresan (1981) demonstrate that futures prices are different from forward prices because of the marking-to-market feature of the futures when interest rates are stochastic. Meulbroek (1992) and Dezbakhsh (1994) showed that the forward-futures price differential can be substantial for financial assets such as Eurodollars and currencies.

7. The expressions for hedge ratios $h_{3F}$ and $h_{3B}$ are both presented below in equations (28) and (29), respectively.

8. Therefore, wherever we could, we will use the same notation as in Chang, Chang and Fang (1996b).

9. The Cox, Ingersoll and Ross (1985) square root process implies that the interest rate cannot be negative. Processes with a Gaussian volatility structure, such as the Ornstein-Uhlenbeck process, allow interest rates to be negative with a (small) positive probability, while processes with a lognormal volatility structure, such as the Black and Karasinski (1991) model, could imply infinite interest rates with positive probability.

10. We follow here the notation of Cox, Ingersoll and Ross (1985) and Chang, Chang and Fang (1996b). The market risk parameter of interest rates, p, will generally collapse to zero.

11. Namely equations (16) and (17) in Chang, Chang and Fang (1996b).

12. We could have presented (30) and (31) differently, for example by showing more distinctly the volatility term of the value of the default-free bond. However, we tried to keep our presentation as close as possible to that of Chang, Chang and Fang (1996b).

13. The reward to risk ratio is defined as the ratio of expected return over the standard deviation of return.

14. Unless $a_S$ and $s_S$ are functions of S, *i* and *t*, and $a_F$ and $s_F$ are functions of *F*, *i* and *t*, as suggested by Chang, Chang and Fang (1996b) in their equations (1) and (2). However, their later treatment of $a_S$, $s_S$, $a_F$ and $s_F$ imply that those are not dependent on S, F, *i* or t.

15. This is equation (19) in Chang, Chang and Fang (1996b).

16. That difference results in the two hedge ratios being of different signs, as illustrated in our numerical example, presented in Table 6 and discussed below.

17. Namely equation (20) in Chang, Chang and Fang (1996b), corrected for some typographical errors.

18. This is equation (21) in Chang, Chang and Fang (1996b).

19. This is equation (22) in Chang, Chang and Fang (1996b).

20. All data are obtained from Bloomberg.

21. Hedging horizon refers to the period over which the investor intends to cover his or her cash position against the asset price risk. We suppose that the hedging horizon coincides with the time to maturity of the futures contract.

22. Chang, Chang and Fang (1996b) used data on returns in their empirical procedures. Yet, their equations (1) and (2), respectively equivalent to our equations (3) and (4), model absolute spot and futures price changes as:

$$dS = \alpha_S\, dt + \sigma_S\, dz_S \qquad (1)$$

$$dF = \alpha_F\, dt + \sigma_F\, dz_F \qquad (2)$$

23. For example, in 1993 the fraction of annual production hedged by these firms varies from zero to 97.5%.