Assurances et gestion des risques
Insurance and Risk Management

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Volume 77, Number 3-4, 2009–2010

URI: https://id.erudit.org/iderudit/1091946ar
DOI: https://doi.org/10.7202/1091946ar

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Publisher(s)

Faculté des sciences de l'administration, Université Laval

ISSN

1705-7299 (print)
2371-4913 (digital)

Cite this article

Boyer, M. & Marquis, O. (2009). A Note on the Valuation of a CDO and of an nth-to-Default CDS Without Monte Carlo Simulation. *Assurances et gestion des risques / Insurance and Risk Management*, *77*(3-4), 265–272. https://doi.org/10.7202/1091946ar

Article abstract

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Assurances et gestion des risques, vol. 77(3-4), octobre 2009-janvier 2010, 265-272
Insurance and Risk Management, vol. 77(3-4), October 2009-January 2010, 265-272

A Note on the Valuation of a CDO and of an n^{th}-to-Default CDS Without Monte Carlo Simulation*

by Martin Boyer and Olivier Marquis

ABSTRACT

The quasi-closed form solution to estimate default probabilities for a group of *n* assets developped in Hull and White (2004) allows a bypass of traditional Monte-Carlo simulations to value credit default swaps and collateralized debt obligations. The authors then construct a simple example to illustrate their approach. The purpose of the present note is to raise the issue that the closed form solution presented does not yield the empirical results of the simple example. Finally, we show how the two can be reconciled.

RÉSUMÉ

La solution quasi-fermée développée dans Hull and White (2004) pour évaluer les probabilités de défaut dans un portefeuille formé de *n* actifs permet de contourner les simulations de Monte-Carlo traditionnelles en vue d'évaluer les options de crédit (Credit Default Swaps) et les titres adossé à des actifs avec flux groupés (ou Collateralized Debt Obligations, CDO). Dans cette optique, les auteurs construisent un exemple simple illustrant cette approche. Le but de notre article est de soulever la question de l'adéquation exacte entre les résultats empiriques de l'exemple simple et la solution quasi-fermée proposée. Nous montrons comment concilier la solution quasi-fermée et les résultats empiriques.

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* We thank Bryan Campbell, Elie Elkhal, Nicolas Papageorgiou and Bruno Rémillard for helpful discussions. The first author thanks the Social Science and Humanities Research Council of Canada (SSHRC) for financial support, and the second author thanks IFM2. All remaining errors are our own.

1. INTRODUCTION

Hull and White (2004) develop a quasi-closed form solution to estimate default probabilities for a group of n assets. This technique allows a bypass of traditional Monte-Carlo simulations to value credit default swaps and collateralized debt obligations. The purpose of the present note is to raise the issue that the closed form solution presented does not yield the empirical results of the example they provide. In this note, we show how to modify the Hull and White model so that the empirical results presented in Hull and White (2004) as Tables 1 and 2 hold.

2. MODEL SETUP

Let us start by summarizing quickly the Gaussian copula technique used by Hull and White (2004), HW04 hereafter. This Gaussian copula is used to model the correlation between the different assets so that there the assets' default probabilities are dependent of one another. Consider the following parameters and variables:

N is the number of assets in the collateralized debt obligation (CDO);

t_i is the time before default of asset i;

$Q_i(t)$ is the risk neutral probability that asset i will default before time t, $P(t_i < t)$;

$S_i(t) = 1 - Q_i(t)$ is the risk neutral probability that the asset will not default before time t;

M is a random variable common to all x_i with mean zero and variance one;

Z_i is an independent random variable with mean zero and variance one;

x_i is a random variable used to build the one-factor copula model in order in incorporate correlation between the assets of the CDO, $x_i = a_i M + \sqrt{1-a_i^2} Z_i$ and $Z_i = \dfrac{x_i - a_i M}{\sqrt{1-a_i^2}}$ for $1 \leq i \leq N$;

$a_i a_j$ is thus the correlation between assets x_i and x_j

HW04 then use this setup to obtain equation (5) of their paper. Equation (5) in HW04 must then be integrated numerically according to the distribution of common factor M. Because M and Z_i are assumed to follow a Gaussian distribution, the numerical integration must be based on the following equation:

$$\pi_t(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N} \left[1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{F_i^{-1}\left(1-e^{-\lambda_i t}\right)-a_i M}{\sqrt{1-a_i^2}}} e^{\left(-\frac{y^2}{2}\right)} dy \right] \sum \left(w_{p(1)} w_{p(2)} \ldots w_{p(k)} \right) \right]$$

$$e^{\left(-\frac{m^2}{2}\right)} dm$$

where

$$W_i = \frac{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{F_i^{-1}\left(1-e^{-\lambda_i t}\right)-a_i M}{\sqrt{1-a_i^2}}} e^{\left(-\frac{y^2}{2}\right)} dy}{1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{F_i^{-1}\left(1-e^{-\lambda_i t}\right)-a_i M}{\sqrt{1-a_i^2}}} e^{\left(-\frac{y^2}{2}\right)} dy}$$

This formula allows us to find the default probabilities given an end period T and a number n of assets in the portfolio. This yields a $n + 1$ x T default matrix. Using a numerical integral approach to evaluate this equation works well when the number of assets is relatively small. When the number of assets is large, stability problems arise so that a recursive method must be used to increase the stability of the results; HW04 present such a method in Appendix A of their paper.

3. VALUING AN NTH TO DEFAULT CREDIT DEFAULT SWAP

We want to replicate Tables 1 and 2 of HW04. To do so, we use their example of a credit default swap (CDS) "in arrears" with quarterly payments that comes to maturity in five years. There are ten assets in the portfolio and their recovery rate is 40%. We also use a constant interest rate term structure of 5% as in HW04. The probability of default follows a Poisson process with mean and variance λ_i for $1 \le i \le 10$. λ_i thus represents default intensity so that

$S_i(t) = e^{-\lambda_i t}$

$Q_i(t) = 1 - e^{-\lambda_i t}$

The CDS equation used in HW04 comes from equation 9 in Hull and White (2000) and equation 5 in Hull and White (2003). The same equation is used in Singh (2003). Consider the following parameters:

T is the maturity of the CDS in years;

R is the recovery rate in a risk neutral world;

$A(t)$ are the accrued interest on the reference bond;

$q(t)$ is the risk neutral probability of a default event at time t;

$v(t)$ is the present value of one dollar at time t;

$u(t)$ is the present value of one dollar received each year between today and time t;

ε (t)$ is the present value of the accrued interest;

π is the risk neutral probability that no default will occur before

maturity, $\pi = 1 - q\,(T)$;

θ is the number of payments (periods) in a year.

The CDS equation is

$$s = \frac{\int_0^T [1 - R - A(t)R]\, q(t) v(t)\, dt}{\int_0^T q(t)[u(t) + \varepsilon(t)]\, dt + \pi u(T)}$$

As stated in Hull and White (2003), it is usually assumed that $A(t) = \varepsilon(t) = 0$ because the cheapest to deliver bond in the event of a default is the bond with the lowest accrued interest. The CDS equation then becomes

$$s = \frac{\int_0^T [1 - R]\, q(t) v(t)\, dt}{\int_0^T q(t)[u(t)]\, dt + \pi u(T)}$$

These integrals are difficult to evaluate so that a numerical integral is used instead. Letting $\pi = (1 - q\,(T))$, the credit default swap spread then becomes

$$s = \frac{\sum_{t=1}^{T*\theta} [1 - R]\, q(t) v(t) \Delta t}{\sum_{t=1}^{T*\theta} q(t) u(t) \Delta t + \left(1 - q(T)\right) u(T)} \qquad \text{(HW)}$$

where the integral has been substituted by a summation over T x θ regions. Using this equation we are not able to replicate the HW04 results with ten equally weighted assets in the portfolio. To illustrate, we present in Tables 1.1 and 2.1 in our first appendix the original HW04 results and compare them to our own simulations in Tables 1.2 and 2.2.

For the first column of Table 1.1, we set $\lambda_i = 0.01$ so that the probability of default for a given asset is 1% each year. We also set $a_i = \sqrt{0.3}$ and M et Z_i normally distributed. For the other two columns, we set $\lambda_i = 0.02$, and $\lambda_i = 0.03$ respectively while keeping $a_i = \sqrt{0.3}$. In Table 2.1, λ_i is set to 0.01 in the three columns. We then let $a_i = 0$, $a_i = \sqrt{0.3}$, and $a_i = \sqrt{0.6}$ in the three columns respectively without changing the 1% default probability.

Tables 1.2 and 2.2 present the results that we obtain using the methodology elaborated by HW04[1]. We see that the differences are sizeable. The differences range between 15% to 50% of the results reported in HW04; interestingly, our results always yield a greater spread. We also note that the differences are larger when the default intensity (λ) is high.

We found a way to replicate the HW04 empirical results when we modify the equation HW as

$$s = \frac{\sum_{t=1}^{T*\theta} [1-R] q(t) e^{\left(-r\left(t+\frac{T\theta}{2}\right)\right)} \Delta t}{\sum_{t=1}^{T*\theta} q(t) e^{\left(-r\left(t+\frac{T\theta+1}{2}\right)\right)} \Delta t + \left(1-q(T)\right) u(T)} \tag{BM}$$

There are two differences between equations HW and BM. First, we note that the numerator of equation BM includes the term $v(t) = \exp(-rt)\exp(-rT\theta/2)$ whereas the numerator of equation HW the term $v(t) = \exp(-rt)$. Second, on the denominator side, equation BM sets $u(t) = \exp(-rt)\exp(-r(T\theta + 1)/2)$ whereas equation HW sets $u(t) = \exp(-rt)$. $u(T)$ is the same in the two equations. We can then rewrite equation BM as

$$s = \frac{v\left(\frac{T\theta}{2}\right)\sum_{t=1}^{T*\theta} [1-R] q(t) v(t) \Delta t}{v\left(\frac{T\theta+1}{2}\right)\sum_{t=1}^{T*\theta} q(t) u(t) \Delta t + \left(1-q(T)\right) u(T)}$$

Letting $\Delta t = 1$, this new formula allows us to find the empirical results displayed in Tables 1.3 and 2.3 in our second appendix. We see that they are the same as those of HW04; the odd one basis point differences are probably caused by rounding errors.

4. DISCUSSION

The purpose of this note was to present an interesting discrepancy in the theoretical model compared to the empirical solution presented in their Hull and White (2004). Our note does not allow us to say that the authors' model is wrong, just that the simulation does not appear to match the specifications of the model. In particular, we do not understand why we need to multiply the numerator by $v\left(\frac{T\theta}{2}\right)$ and the first term of the denominator by $v\left(\frac{T\theta+1}{2}\right)$ to obtain Hull and White's empirical results.

It is also possible that our interpretation of the Hull and White (2004) model is erroneous. For instance, it is possible that we do not use the same numerical integral procedure as HW04 (rectangular versus trapezoid for example). We doubt, however, that this would explain the 35% difference (176 basis points) in the 2nd to default credit default swap when $\lambda = 0.03$. Nevertheless, even if our interpretation of the HW04 model is erroneous, our paper still provides a different approach that yields the same spreads.

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Note

1. These results were obtained using Matlab.

APPENDIX 1
COMPARISON BETWEEN THE HULL AND WHITE RESULTS AND OURS USING THEIR MODEL

Hull and White (2004)			
Table 1.1*			
	Default intensity for all firms		
n	0.01	0.002	0.03
1	440	814	1165
2	139	321	513
3	53	149	263
4	21	71	139
5	8	34	72
6	3	15	36
7	1	6	16
8	0	2	6
9	0	1	2
10	0	0	0

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Table 1.2*			
	Default intensity for all firms		
n	0.01	0.002	0.03
1	563	1151	1803
2	165	405	689
3	61	178	329
4	24	83	167
5	10	39	85
6	4	17	41
7	1	7	19
8	0	3	7
9	0	1	2
10	0	0	0

* Spread to buy 5 year protection for the nth default from a basket of 10 names. All firms have the same probability of default. The correlaton between each pair of names is 0.3. The spread is in basis point per annum.

Table 2.1**			
	Pairwise correlation		
n	0	0.3	0.6
1	603	440	293
2	98	139	137
3	12	53	79
4	1	21	49
5	0	8	31
6	0	3	19
7	0	1	12
8	0	0	7
9	0	0	3
10	0	0	1

Table 2.2**			
	Pairwise correlation		
n	0	0.3	0.6
1	818	563	358
2	116	165	162
3	14	61	92
4	1	24	56
5	0	10	35
6	0	4	22
7	0	1	13
8	0	0	7
9	0	0	4
10	0	0	1

** Spread to buy 5 year protection for the n^{th} default from a basket of 10 names. All pairs of firms have the same correlation. The default intensity for each firm is 0.01. The spread is in basis point per annum.

APPENDIX II
COMPARISON BETWEEN THE HULL AND WHITE RESULTS AND OURS USING THEIR MODEL

Hull and White (2004)			
Table 1.1*			
	Default intensity for all firms		
n	0.01	0.002	0.03
1	440	814	1165
2	139	321	513
3	53	149	263
4	21	71	139
5	8	34	72
6	3	15	36
7	1	6	16
8	0	2	6
9	0	1	2
10	0	0	0

Boyer and Marquis II			
Table 1.3*			
	Default intensity for all firms		
n	0.01	0.002	0.03
1	439	814	1165
2	139	321	514
3	53	149	264
4	21	72	140
5	8	34	73
6	3	15	36
7	1	6	16
8	0	2	7
9	0	1	2
10	0	0	0

* Spread to buy 5 year protection for the nth default from a basket of 10 names. All firms have the same probability of default. The correlaton between each pair of names is 0.3. The spread is in basis point per annum.

Table 2.1**			
	Pairwise correlation		
n	0	0.3	0.6
1	603	440	293
2	98	139	137
3	12	53	79
4	1	21	49
5	0	8	31
6	0	3	19
7	0	1	12
8	0	0	7
9	0	0	3
10	0	0	1

Table 2.3**			
	Pairwise correlation		
n	0	0.3	0.6
1	603	439	292
2	98	139	137
3	12	53	79
4	1	21	49
5	0	8	31
6	0	3	19
7	0	1	12
8	0	0	7
9	0	0	3
10	0	0	1

** Spread to buy 5 year protection for the n^{th} default from a basket of 10 names. All pairs of firms have the same correlation. The default intensity for each firm is 0.01. The spread is in basis point per annum.