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Article abstract

In recent years, the magnitudes of realized catastrophe (extreme-event) losses have increased dramatically. The effects of increasing catastrophe risks on the insurance industry have been profound. In the current private insurance market, the possibility of insurer default is of great concern to insurers and their investors. However, there is limited actuarial or financial theory for analyzing catastrophe insurance contracts based upon the probability of ruin. In this article, we develop a mixed diffusion and compound Poisson jump model of insurer net worth to reflect the fact that insurers are faced with both non-catastrophe and catastrophe risks. Under the assumption of exponentially distributed catastrophe losses, we derive analytical approximations to the insurer ruin probability. Assuming constant catastrophe loss amounts, we calculate the ruin probability numerically and compare the results with those for exponentially distributed losses.

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ABSTRACT

In recent years, the magnitudes of realized catastrophe (extreme-event) losses have increased dramatically. The effects of increasing catastrophe risks on the insurance industry have been profound. In the current private insurance market, the possibility of insurer default is of great concern to insurers and their investors. However, there is limited actuarial or financial theory for analyzing catastrophe insurance contracts based upon the probability of ruin. In this article, we develop a mixed diffusion and compound Poisson jump model of insurer net worth to reflect the fact that insurers are faced with both non-catastrophe and catastrophe risks. Under the assumption of exponentially distributed catastrophe losses, we derive analytical approximations to the insurer ruin probability. Assuming constant catastrophe loss amounts, we calculate the ruin probability numerically and compare the results with those for exponentially distributed losses.

Keywords : catastrophe risk, reinsurance, insurer solvency, diffusion-jump analysis.

RÉSUMÉ

Au cours des années récentes, les catastrophes réalisées ont beaucoup augmenté. Les effets de l'augmentation des risques de catastrophe sur l'industrie de l'assurance ont été profonds. Dans les marchés d'assurance courants, la possibilité qu'un assureur fasse faillite est une source importante d'incertitude pour les assureurs et leurs investisseurs. Par contre, la théorie financière et actuarielle est limitée pour analyser les contrats d'assurance avec risque de catastrophe basés sur une probabilité de défaillance. Dans cet article, nous développons un processus de diffusion basé sur un modèle Poisson qui tient compte du fait que l'assureur est confronté à la fois à du risque de catastrophe et de non-catastrophe. Sous l'hypothèse que les pertes de catastrophe sont distribuées de façon exponentielle, nous obtenons une forme analytique de la probabilité de défaillance de l'assureur. Sous l'hypothèse que les pertes de catastrophe sont constantes, nous calculons la probabilité de défaillance numériquement et comparons les résultats à ceux sous l'hypothèse qu'elles sont distribuées de façon exponentielle.

Mots clés : risque catastrophique, réassurance, solvabilité des assureurs, diffusion par sauts.

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INTRODUCTION

In recent years, the magnitudes of realized catastrophe (extreme-event) losses have increased dramatically. Hurricane Andrew in 1992 and the Northridge earthquake in 1994 caused insured losses of approximately \$20 billion and \$16 billion, respectively—sizes unthinkable before the 1990s. The September 11 terrorism attacks in 2001 caused simultaneous losses over multiple lines of business, including property, business-interruption, workers' compensation, and life insurance. Estimates of insured loss amounts from these attacks range from \$50 to \$70 billion.

The effects of increasing catastrophe risks on the insurance industry have been profound. Lewis and Murdock (1996) contend that because of the infrequency and large magnitude of extreme-event losses, catastrophe risks need to be diversified intertemporally as well as spatially. Given that catastrophe losses are geographically correlated within a risk pool, intertemporal diversification is clearly important. However, the possibility of insurer bankruptcy impedes the successful time-diversification of large losses, because asymmetry and high capital costs often preclude insurers from maintaining sufficient capital (net worth) to guarantee the risk pool against bankruptcy. Froot and O'Connell (1999) show that realized catastrophe losses lead to higher prices for and reduced supply of catastrophe reinsurance because of capital market imperfections.

Since the U.S. federal government effectively has zero default probability, it can borrow at the risk-free rate, and is thus in a unique position to diversify claims intertemporally. For this reason, Lewis and Murdock (1996) proposed a market-based model in which the federal government would attempt to extend private insurance capacity though the creation of a federal excess-of-loss (XOL) reinsurance mechanism narrowly targeted to the missing market for the intertemporal diversification of catastrophe losses. Assuming that the XOL reinsurance contracts can cure (or greatly alleviate) market imperfection, Cummins, Lewis, and Phillips (1999) provide a pricing formula for them.

In the current private insurance market, the possibility of insurer default is of great concern to investors and insurers. However, there is limited actuarial or financial theory for analyzing catastrophe insurance contracts based upon the probability of ruin. In this article, we develop a mixed diffusion and compound Poisson jump model of insurer net worth to reflect the fact that insurers are faced with both non-catastrophe and catastrophe risks. Under the assumption of exponentially distributed catastrophe losses, we derive analytical approximations to the insurer ruin probability. Assuming constant catastrophe loss amounts, we calculate the ruin probability numerically and compare the results with those for exponentially distributed losses. All proofs are in the Appendix.

MODELS OF INSURER NET WORTH

The classical actuarial model of insurer net worth,

$$U(t) = U(0) + ct - \sum_{i=1}^{N(t)} X_i, \tag{1}$$

(see, e.g., Bowers, et al., 1997, p. 399) is based upon assumptions of linear income and a compound Poisson jump process. However, when the number of claims is large, and the size of losses is small relative to the insurer net worth, the insurer's net worth essentially becomes a diffusion (Iglehart, 1969). Using diffusion models, it is possible to incorporate stochastic premium and investment income rates into the analysis of insurer net worth, as well as to study first-passage times to ruin analytically (see, for example, Schmidli, 1994 and Powers, 1995).

Diffusion models are based upon the assumption that the net worth process possesses a continuous sample path with probability one. This means that in a short interval of time net worth can change by only a very small amount. While such models are entirely reasonable for non-catastrophe insurance risk, they do not permit the modeling of catastrophe risk, which exposes insurers to potentially large jumps in liabilities.

Insurers need a model that considers both non-catastrophe and catastrophe risks simultaneously. However, the theoretical literature on this topic is fairly limited. Merton (1976) developed an option pricing formula for the case in which stock returns are generated by a mixture of both continuous and jump processes. In that model, the continuous process reflects the "normal" variation in price that causes marginal changes in stock value, whereas the jump process describes the "abnormal" variation in price that has more than a marginal effect on value. Cummins (1988) extended Merton's theory, developing a pricing model for an insurance guaranty fund when insurers are subjected to both continuous (non-catastrophe) and discontinuous (catastrophe) changes in liabilities.

Both Merton (1976) and Cummins (1988) used equilibrium models assuming that the market is "frictionless", the jump risk is nonsystematic (uncorrelated with the market), and the Capital Asset Pricing Model (CAPM) is a valid description of equilibrium security/insurance price. These assumptions lead to the conclusion that the required return for bearing catastrophe risk is simply the risk-free rate, and so the fair premium on a catastrophe insurance contract should be less than or equal to the actuarial value of the contract loss. However, in today's insurance market, catastrophe insurance premiums are considerably greater than actuarial estimates of expected losses.

One significant reason why catastrophe insurance prices are high is that the capital market for catastrophe risk-taking is limited (Froot, 1999). Without sufficient capital (net worth), insurers are faced with a greater risk of insolvency. This default risk imposes both explicit and implicit cost on insurance firms (Hoerger, Sloan, and Hassan, 1990), including changes in company bond ratings, regulatory intervention, and restructuring. The possibility of insolvency also prevents insurers from effectively diversifying disaster risks intertemporally (Lewis and Murdock, 1996). Consequently, it is important to consider the probability of ruin in studying the price and supply of catastrophe insurance contracts.

In this article, we develop a mixed diffusion and jump model of insurer net worth, in which the change in net worth due to underwriting non-catastrophe risk is described by a Brownian motion with drift, and the change in net worth due to underwriting catastrophe risk is described by a classical compound Poisson jump process. This model provides a different interpretation of the classical risk process "perturbed by diffusion" of (Dufresne and Gerber, 1991). In the classical process perturbed by diffusion, the principal insurance risks are described by a compound Poisson loss process, and the diffusion process simply provides an additional layer of risk with regard to either losses or premium income.

Under the mixed diffusion-jump model, we derive analytical approximations to the insurer's probability of ruin when the magnitude of catastrophe losses is exponentially distributed. These results show analytically how underwriting catastrophe risk affects an insurer's ruin probability, and provide insurers an effective tool for making such underwriting decisions.

Since most individual insurers write only one specific layer of catastrophe risk at a time, it is reasonable (and conservative) to assume that the magnitude of each catastrophe loss is the (constant) upper limit of the layer. Under this assumption, we develop numerical solutions for the insurer ruin probability when loss amounts are constant, and compare these results to those for the exponential case. Not surprisingly, the comparison shows that for most insurers, the assumption of exponential losses is more conservative than the assumption of constant losses.

THE DIFFUSION-JUMP MODEL OF INSURER NET WORTH

Diffusion Processes and Jump Processes

For $t \geq 0$, let $U(t)$ denote an insurer's net worth at time t, and let $U(0) = u_0 > 0$ be the insurer's initial net worth. We use the notation $U_D(t)$ to denote a net worth model described by a simple diffusion process with Brownian motion and positive drift, and $U_P(t)$ to denote a model described by the classical model (1) with linear income and compound Poisson losses.

For the diffusion case, we model the change in $U_D(t)$ due to underwriting non-catastrophe risk and the accumulation of investment income by the stochastic differential equation (SDE)

$$dU_D(t) = \alpha dt + \sqrt{\beta} dZ(t), \tag{2}$$

where : α represents the instantaneous growth of insurer net worth due to underwriting non-catastrophe risk and the accumulation of investment income; β is the instantaneous variance of the insurer net worth, given no catastrophe events; and $Z(t)$ is a standard Brownian motion representing the local disturbance caused by non-catastrophe insurance operations. We assume that once an insurer's net worth reaches zero, the insurer becomes insolvent, and cannot recover. Therefore, $U_D(t)$ has an absorbing barrier at $U_D(t) = 0$, and we denote the probability of ruin by

$$\psi_D(u_0) = \Pr\{\exists\, t \in (0,\infty) \text{ s.t. } U_D(t) \leq 0 | U_D(0) = u_0\}.$$

The classical jump process (1) can be characterized by the SDE

$$dU_P(t) = (1+\theta)\lambda\mu dt - X(t)dN(t), \tag{3}$$

where $N(t) \in \{0,1,2,\ldots\}$ denotes a Poisson counting process with parameter λt, the $X(t) \in [0,\infty)$ are i.i.d. random variables independent of $N(t)$ with mean μ, and θ is the insurer's total profit loading. The process $U_P(t)$ also has an absorbing barrier at $U_P(t) = 0$, and the probability of ruin is denoted by

$$\psi_P(u_0) = \Pr\{\exists\, t < (0,\infty) \text{ s.t. } U_P(t) \le 0 | U_P(0) = u_0\}.$$

☐ The Mixed Process

Combining the processes described by equations (2) and (3), we model the insurer's net worth by a mixed continuous and jump process subject to the SDE

$$dU_M(t) = [\alpha + (1+\theta)\lambda\mu]dt + \sqrt{\beta}dZ(t) - X(t)dN(t), \quad (4)$$

and denote the probability of ruin by

$$\psi_M(u_0) = \Pr\{\exists\, t < (0,\infty) \text{ s.t. } U_M(t) \le 0 | U_M(0) = u_0\}.$$

In the spirit of Powers (2002), we define a portfolio of catastrophe (extreme-event) risk as a collection of exposures during the time period $(0,t)$ that produces the total loss amount

$$L(t) = \int_0^t X(\tau)dN(\tau),$$

where $E[N(t)] = \lambda = O(h)t$, and $E[X(t)] = \mu = O(1/h)$, for some small real number $h > 0$.[1] We study $U_M(t)$ as $h \to 0$ and $\lambda\mu \to \pi \in (0,\infty)$ for two cases : (1) $u_0 = \kappa\mu + O(h)$, and (2) u_0 is constant.

We conclude this section by stating three basic results regarding the separate diffusion and jump processes that provide a useful context for our further analysis of the mixed diffusion-jump process in a later section.

Lemma 1 : If an insurer's net worth process $U_D(t)$ is described by the SDE (2), then :

(i) $\psi_D(u_0) = e^{-\frac{2\alpha u_0}{\beta}}$;

(ii) for $u_0 = \kappa\mu + O(h)$, $\lim_{h\to 0} \psi_D(u_0) = 0$; and

(iii) for fixed u_0, $\lim_{h\to 0} \psi_D(u_0) = e^{-\frac{2\alpha u_0}{\beta}}$.

Lemma 2 : If an insurer's net worth process $U_P(t)$ is described by the SDE (3) such that the $X(t)$ are i.i.d. exponential random variables with mean μ, then:

(i) $\psi_P(u_0) = \frac{1}{1+\theta} e^{-\frac{\theta u_0}{(1+\theta)\mu}}$;

(ii) for $u_0 = \kappa\mu + O(h)$, $\lim_{h \to 0} \psi_P(u_0) = \frac{1}{1+\theta} e^{-\frac{\theta\kappa}{1+\theta}}$; and

(iii) for fixed u_0, $\lim_{h \to 0} \psi_P(u_0) = \frac{1}{1+\theta}$.

Lemma 3 : If an insurer's net worth process $U_p(t)$ is described by the SDE (3) such that $X(t) = \mu$ for all t, then :

$$
\text{(i) } \psi_P(u_0) = \begin{cases} \frac{1}{1+\theta} \text{ for } u_0 = 0 \\ 1 - \frac{\theta}{1+\theta} e^{\frac{1}{1+\theta}} \text{ for } u_0 = \mu \\ 1 - \left[1 - \psi_P(u_0 - \mu)\right] e^{\frac{1}{1+\theta}} \\ + \sum_{j=1}^{\frac{u_0}{\mu}-1} \frac{1}{j!} \left(\frac{1}{1+\theta}\right)^j \left[1 - \psi_P(u_0 - j\mu)\right] \\ \text{for } u_0 = \kappa\mu,\ \kappa \in \{2,3,4,\ldots\} \end{cases};
$$

(ii) for $u_0 = \kappa\mu + O(h)$,

$$
\lim_{h \to 0} \psi_P(u_0) = \begin{cases} \frac{1}{1+\theta} \text{ for } \kappa = 0 \\ 1 - \frac{\theta}{1+\theta} e^{\frac{1}{1+\theta}} \text{ for } \kappa = 1 \\ 1 - \left[1 - \psi_P((\kappa-1)\mu)\right] e^{\frac{1}{1+\theta}} \\ + \sum_{j=1}^{\kappa-1} \frac{1}{j!} \left(\frac{1}{1+\theta}\right)^j \left[1 - \psi_P((\kappa-j)\mu)\right] \\ \text{for } \kappa \in \{2,3,4,\ldots\} \end{cases}; \text{ and}
$$

(iii) for fixed u_0, $\lim_{h \to 0} \psi_P(u_0) = \frac{1}{1+\theta}$.

THE MIXED PROCESS WITH EXPONENTIAL CATASTROPHE LOSS AMOUNTS

The Probability of Ruin

The following theorem provides analytical approximations to the probability of ruin for $U_M(t)$ under two different assumptions regarding the insurer's initial net worth.

Theorem 1 : If an insurer's net worth process $U_M(t)$ is described by the SDE (4) such that the $X(t)$ are i.i.d. exponential random variables with mean μ, then:

(i) for $u_0 = \kappa\mu + O(h)$

$$\hat{\psi}_M(u_0) = \lim_{h \to 0} \psi_M(u_0) = \frac{\pi}{\alpha + (1+\theta)\pi} e^{-\left[\frac{\alpha+\theta\pi}{\alpha+(1+\theta)\pi}\right]\kappa}; \text{ and}$$

(ii) for fixed u_0,

$$\hat{\psi}_M(u_0) = \lim_{h \to 0} \psi_M(u_0) = \frac{\alpha + \theta\pi}{\alpha + (1+\theta)\pi} e^{-\frac{2[\alpha+(1+\theta)\pi]u_0}{\beta}} + \frac{\pi}{\alpha + (1+\theta)\pi}.$$

Proof : See the Appendix

The Marginal Addition of Catastrophe Risk

The results of Theorem 1 can be applied immediately to the study of how the incorporation of catastrophe risk into an insurer's portfolio will change the insurer's probability of ruin. In this sub-section, we consider the case of adding a catastrophe risk process on the margin, without reducing the non-catastrophe component of the insurer's portfolio. In the following sub-section, we consider the case of substituting catastrophe risk for non-catastrophe risk.

Let

$$dU_A(t) = [\alpha + (1+\theta)w\lambda\mu]dt + \sqrt{\beta}dZ(t) - X(t)dN_w(t), \qquad (5)$$

where $N_w(t) \sim \text{Poisson}(w\lambda t)$. In other words, let $U_A(t)$denote a net worth process in which the introduction of catastrophe risk is governed by the parameter w in the Poisson counting process. We then obtain the following results.

Theorem 2 : If an insurer's net worth process is described by the SDE (5) such that the $X(t)$ are i.i.d. exponential random variables with mean μ, then :

(i) for $u_0 = \kappa\mu + O(h)$, the *addition* of catastrophe risk to a non-catastrophe or mixed portfolio never enhances the insurer's solvency; and

(ii) for fixed u_0, the *addition* of catastrophe risk to a non-catastrophe portfolio enhances the insurer's solvency if and only if

$$\theta > \frac{\left(e^{\frac{2\alpha u_0}{\beta}} - 1 - \frac{2\alpha u_0}{\beta}\right)}{\frac{2\alpha u_0}{\beta}} = \sum_{i=1}^{\infty} \frac{1}{(i+1)!}\left(\frac{2\alpha u_0}{\beta}\right)^i.$$

Proof : See the Appendix

Theorem 2 provides an interesting and somewhat surprising insight. By comparing the results of parts (i) and (ii), we see that the insurer can improve its solvency (reduce its ruin probability) by adding catastrophe risk when its level of net worth is dwarfed by the magnitude of individual catastrophe losses—i.e., when u_0 is constant as $\mu \to \infty$. However, the insurer cannot improve its solvency by adding catastrophe risk when u_0 is of the same order of magnitude as the expected loss amount—i.e., when $u_0 = \kappa\mu + O(h)$.

This result appears counterintuitive : after all, one would think that the potential for enhancing solvency through the addition of catastrophe risk would be greater when u_0 is more substantial compared to μ. However, the result can be explained by noting that when $u_0 = \kappa\mu + O(h)$, the ruin probability is already relatively small, and cannot be improved simply by adding capital through premium income, regardless of the size of the profit loading θ. The opposite situation holds for the case in which u_0 is constant; in fact, part (ii) shows that adding catastrophe risk to an insurer's portfolio is helpful when the profit loading θ is large and/or the insurer's intrinsic level of risk is great (i.e., $\frac{2\alpha u_0}{\beta}$ is small). Part (ii) also provides the following simple rule-of-thumb for selecting θ (when $\frac{2\alpha u_0}{\beta} < 1$) :

$$\theta > \frac{\alpha u_0}{\beta}.$$

In short, underwriting catastrophe risk may reduce the probability of ruin for high-risk companies (i.e., those with an initially high ruin probability), but can only increase the probability of ruin for safer companies (i.e., those with an initially low ruin probability).

☐ The Marginal Substitution of Catastrophe Risk

We now consider the case of substituting catastrophe risk for non-catastrophe risk. Let

$$\begin{aligned} dU_S(t) &= \left[(1-w)\alpha + (1+\theta)w\lambda\mu\right]dt \\ &\quad + (1-w)\sqrt{\beta}\,dZ(t) - X(t)\,dN_w(t), \end{aligned} \tag{6}$$

where $N_w(t) \sim \text{Poisson}(w\lambda t)$. In other words, let $U_S(t)$ denote a net worth process in which the introduction of catastrophe risk is governed by the parameter w in the Poisson counting process, and the reduction of non-catastrophe risk is governed by the complementary parameter $1-w$.

Theorem 3 : If an insurer's net worth process is described by the SDE (6) such that the $X(t)$ are i.i.d. exponential random variables with mean μ, then :

(i) for $u_0 = \kappa\mu + O(h)$, the *substitution* of catastrophe risk into a non-catastrophe or mixed portfolio never enhances the insurer's solvency; and

(ii) for fixed u_0, the *substitution* of catastrophe risk into a non-catastrophe portfolio enhances the insurer's solvency if and only if

$$\theta > \frac{\left(e^{\frac{2\alpha u_0}{\beta}} - 1 - \frac{2\alpha u_0}{\beta}\right)}{\frac{2\alpha u_0}{\beta}} - \frac{\alpha}{\pi} = \sum_{i=1}^{\infty} \frac{1}{(i+1)!}\left(\frac{2\alpha u_0}{\beta}\right)^i - \frac{\alpha}{\pi}.$$

Proof : See the Appendix

The results of Theorem 3 are analogous to those of Theorem 2. Here we see that the insurer can improve its solvency (reduce its ruin probability) by substituting catastrophe risk when its level of net worth is much smaller than the magnitude of individual catastrophe losses—i.e., when u_0 is constant as $\mu \to \infty$. However, the insurer cannot improve its solvency by substituting catastrophe risk when $u_0 = \kappa\mu + O(h)$.

The explanation for this phenomenon is the same as before. When $u_0 = \kappa\mu + O(h)$, the ruin probability is relatively small, and cannot be improved by adding capital through premium income, regardless of the size of the profit loading θ. The opposite holds for the case in which u_0 is constant; in fact, part (ii) shows that *substituting* catastrophe risk for non-catastrophe risk can boost solvency with a smaller profit loading than that which is needed when *adding* catastrophe risk. This is because substitution, unlike addition, does not spread the insurer's net worth over a larger total pool of risk. As in Theorem 2, part (ii) provides a simple rule-of-thumb for selecting θ (when $\frac{2\alpha u_0}{\beta} < 1$):

$$\theta > \alpha\left(\frac{u_0}{\beta} - \frac{1}{\pi}\right).$$

THE MIXED PROCESS WITH CONSTANT CATASTROPHE LOSS AMOUNTS

Theory

In the previous section, we treated the magnitude of catastrophe losses as an exponential random variable. However, one might argue that this assumption is unrealistic because of the highly skewed nature of catastrophe losses. In fact, while it is true that raw catastrophe loss amounts are highly skewed, it is generally the case that individual insurers cover only one layer of catastrophe risk at a time. Therefore, there is generally an upper limit to an insurer's loss payment when a catastrophe occurs, and it is reasonable (and conservative) to assume that whenever a catastrophe loss occurs, the insurer simply pays the upper limit.

Under this assumption, we will compute the insurer's ruin probabilities numerically, and compare the results with those when the magnitude of catastrophe losses is exponentially distributed. We will show that by setting the exponential mean equal to the upper limit of the catastrophe loss payment, the results in the previous section provide a conservative estimate of the insurer's ruin probability.

We begin by showing that when the catastrophe loss amount $X(t)$ is constant, the insurer's probability of ruin satisfies a certain ordinary differential equation (ODE). To simplify notation, let $\gamma = \alpha + (1+\theta)\lambda\mu$, so that the mixed diffusion-jump model given by equation (4) becomes

$$dU_M(t) = \gamma dt + \sqrt{\beta} dZ(t) - X(t) dN(t) \tag{7}$$

Theorem 4 : If an insurer's net worth process is described by the SDE (7) such that $X(t) = \mu$ for all t, then the ultimate survival probability, $\zeta_M(u_0) = 1 - \psi_M(u_0)$, satisfies the ODE

$$0 = \begin{cases} \frac{1}{2}\beta \frac{\partial^2 \zeta_M(u_0)}{\partial u_0^2} + \gamma \frac{\partial \zeta_M(u_0)}{\partial u_0} - \lambda \zeta_M(u_0) + \lambda \zeta_M(u_0 - \mu) \\ \text{for } u_0 \geq \mu \\ \\ \frac{1}{2}\beta \frac{\partial^2 \zeta_M(u_0)}{\partial u_0^2} + \gamma \frac{\partial \zeta_M(u_0)}{\partial u_0} - \lambda \zeta_M(u_0) \text{ for } u_0 < \mu \end{cases}, \tag{8}$$

subject to the boundary conditions $\zeta_M(u_0) = 0$ and $\zeta_M(\infty) = 1$.

Proof : See the Appendix

To study the case of adding catastrophe risk on the margin, we replace λ by $w\lambda$ in the ODE (8), and see that $\zeta_A(u_0) = 1 - \psi_A(u_0)$ must satisfy

$$0 = \begin{cases} \frac{1}{2}\beta \frac{\partial^2 \zeta_A(u_0)}{\partial u_0^2} + \gamma_A \frac{\partial \zeta_A(u_0)}{\partial u_0} - w\lambda \zeta_A(u_0) + w\lambda \zeta_A(u_0 - \mu) \\ \text{for } u_0 \geq \mu \\ \\ \frac{1}{2}\beta \frac{\partial^2 \zeta_A(u_0)}{\partial u_0^2} + \gamma_A \frac{\partial \zeta_A(u_0)}{\partial u_0} - w\lambda \zeta_A(u_0) \text{ for } u_0 < \mu \end{cases} \tag{9}$$

where $\gamma_A = \alpha + (1 + \theta)w\lambda\mu$. Similarly, to study the case of substituting catastrophe risk for non-catastrophe risk on the margin, we replace α by $(1 - w)\alpha$, β by $(1 - w)^2\beta$, and λ by $w\lambda$ in (8), and see that $\zeta_S(u_0) = 1 - \psi_S(u_0)$ must satisfy

$$
0 = \begin{cases} \frac{1}{2}(1-w)^2\beta\frac{\partial^2\zeta_S(u_0)}{\partial u_0^2} + \gamma_S\frac{\partial\zeta_S(u_0)}{\partial u_0} - w\lambda\zeta_S(u_0) \\ +w\lambda\zeta_S(u_0-\mu) \text{ for } u_0 \geq \mu \\ \\ \frac{1}{2}(1-w)^2\beta\frac{\partial^2\zeta_S(u_0)}{\partial u_0^2} + \gamma_S\frac{\partial\zeta_S(u_0)}{\partial u_0} - w\lambda\zeta_S(u_0) \text{ for } u_0 < \mu \end{cases}, \quad (10)
$$

where $\gamma_S = (1-w)\alpha + (1+\theta)w\lambda\mu$. Equations (8), 9), and (10) are ODEs with a shift. Although they are difficult to solve analytically, one can use Euler methods to obtain numerical solutions.

☐ Numerical Results

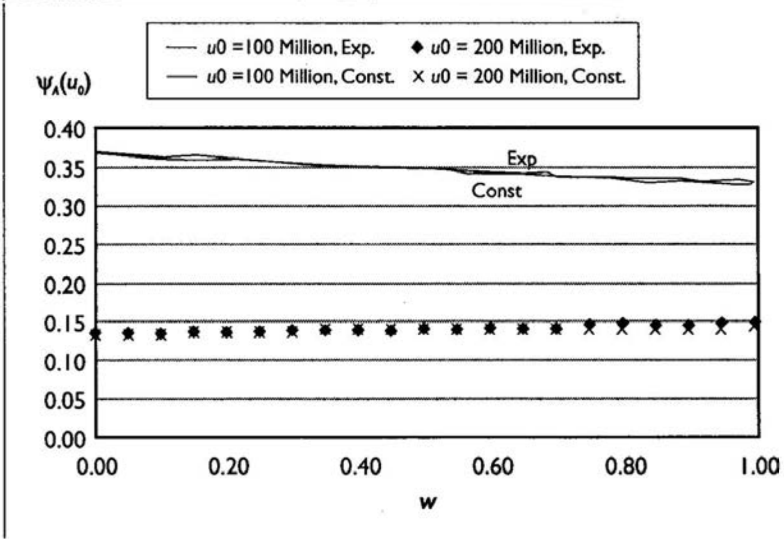
We now provide an analysis of the ruin probabilities for the mixed model with constant loss amounts. Because we plan to compare these results with those for the mixed model with exponential losses, we will assume that u_0 is constant as μ grows large. Then, by Theorems 2(ii) and 3(ii), there will exist some values of θ for which the insurer can benefit from incorporating catastrophe risk with exponential losses into its portfolio.

The following parameter values are used in our analysis: $\alpha = 10^7$, $\beta = 2 \cdot 10^{15}$, $\theta = 2$, $\lambda = 10^{-3}$, $\mu = 10^9$, and $u_0 = 10^8$ or $2 \cdot 10^8$. For the case of exponential loss amounts, these parameters are inserted into the asymptotic equations for $\hat{\psi}_A(u_0)$ and $\hat{\psi}_S(u_0)$.

For the *addition* of catastrophe risk, Figure 1 provides comparisons of ruin probabilities for the mixed model with exponential and constant loss amounts, respectively. From this figure, we immediately observe that the ruin probabilities for the constant loss case are very close to—but consistently less than—those for the exponential loss case. This shows that, for the parameter values selected, the greater variability of the exponential losses translates into a greater probability of ruin, which can thus be used as a conservative (upper) bound on $\psi_A(u_0)$ for the constant loss case.

To provide some context for the various ruin probabilities shown, we note that when , $w = 0$, the mixed process becomes a pure diffusion, and so $\psi_A(u_0) = e^{-\frac{2\alpha u_0}{\beta}}$ by Lemma 1(iii). As $w \to \infty$, on the other hand, the mixed process becomes a classical compound

FIGURE 1
PROBABILITY OF RUIN WHEN ADDING CATASTROPHE RISK



Poisson jump process, and so $\psi_A(u_0) \to \frac{1}{1+\theta}$ by Lemma 2(iii) and Lemma 3(iii).

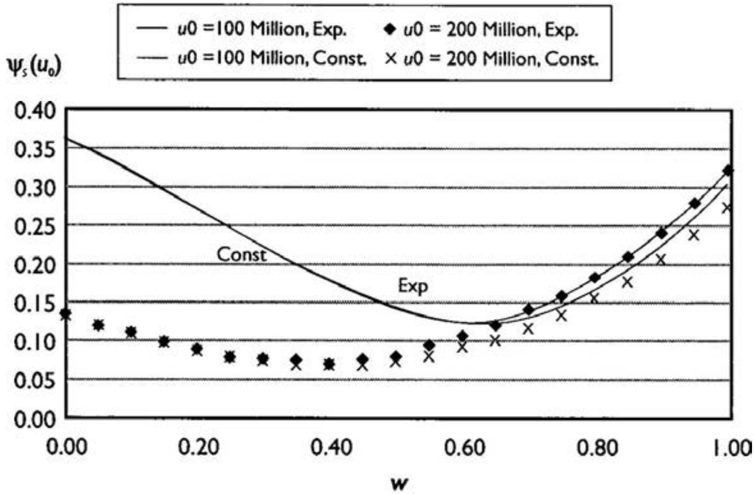
Comparing the ruin probabilities for $u_0 = 10^8$ with those for $u_0 = 2 \cdot 10^8$, we see that for the smaller value of u_0, $\psi_A(u_0)$ is initially decreasing over w (i.e., at $w = 0$), whereas for $u_0 = 2 \cdot 10^8$, $\psi_A(u_0)$ is initially increasing. These two cases correspond to the two alternatives indicated by Theorem 2(ii) : for $u_0 = 10^8$,

$$\theta > \frac{\left(e^{\frac{2\alpha u_0}{\beta}} - 1 - \frac{2\alpha u_0}{\beta}\right)}{\frac{2\alpha u_0}{\beta}}, \text{ whereas for } u_0 = 2 \cdot 10^8,$$

$$\theta < \frac{\left(e^{\frac{2\alpha u_0}{\beta}} - 1 - \frac{2\alpha u_0}{\beta}\right)}{\frac{2\alpha u_0}{\beta}}.$$

We now consider the *substitution* of catastrophe risk for non-catastrophe risk, as shown in Figure 2. Here, we see again that the ruin probabilities for the constant loss case are very close to—but consistently less than—those for the exponential loss case. Thus, for the parameter values selected, the exponential loss assumption provides a conservative (upper) bound on the ruin probabilities.

FIGURE 2
PROBABILITY OF RUIN WHEN SUBSTITUTING CATASTROPHE RISK



Looking from left to right in Figure 2, we see the transition from a pure diffusion (at $w = 0$) to a classical compound Poisson jump process (at $w = 1$). Thus, by Lemma 1(iii), $\psi_S(u_0) = e^{-\frac{2\alpha u_0}{\beta}}$ at $w = 0$, and by Lemma 2(iii) and Lemma 3(iii), $\psi_S(u_0) = \frac{1}{1+\theta}$ at $w = 1$.[2] In a sense, the curves in Figure 2 provide an accelerated view of what happens to the curves in Figure 1 as $w \to \infty$. However, it is important to note that the curves in Figure 2 are not simply "stretched out" isomorphisms of those in Figure 1. Specifically, we note that when $u_0 = 2 \cdot 10^8$, $\psi_S(u_0)$ is initially decreasing over w (i.e., at $w = 0$), whereas $\psi_A(u_0)$ is initially increasing. This difference occurs precisely because the condition given by Theorem 3(ii) is more easily satisfied than is the condition given by Theorem 2(ii).

CONCLUSION

In this study, we have developed a mixed diffusion-jump model of insurer net worth. The diffusion portion consists of a Brownian motion with drift, which represents changes in net worth due to underwriting non-catastrophe risk and the accumulation of investment income; the jump portion consists of a classical compound Poisson jump process, which represents changes in net worth due to underwriting catastrophe risk.

When the magnitude of catastrophe losses is exponentially distributed, we find approximate analytical solutions for the insurer's probability of ruin, and provide conditions under which insurers can enhance solvency by adding catastrophe risk to their portfolios, or by substituting catastrophe risk for non-catastrophe risk.

Given the common insurer practice of covering only one layer of catastrophe risk at a time, it is reasonable and conservative to set the amount of each catastrophe loss equal to the upper limit of the catastrophe insurance contract. Under the assumption of constant loss amounts, we solve for the insurer's probability of ruin numerically, and show that slightly lower ruin probabilities are obtained than in the case of exponential losses. Therefore, by setting the exponential mean equal to the upper limit of an insurer's catastrophe insurance contract, the insurer can conservatively evaluate the effects of incorporating catastrophe risk into its insurance portfolio.

APPENDIX

Proof of Lemma 1 : Part (i) states the well-known probability of passage from u_0 to 0 for a simple Brownian motion with positive drift (see, e.g., Powers, 1995). Parts (ii) and (iii) follow immediately from (i) by taking the appropriate limits.

Proof of Lemma 2 : Part (i) states the well-known probability of ruin for the classical net worth process with linear income and compound Poisson/exponential losses (see, e.g., Bowers, et al., pp. 414-415). Parts (ii) and (iii) follow immediately from (i) by taking the appropriate limits.

Proof of Lemma 3 : The probability of ruin for the classical net worth process with linear income and compound Poisson/constant losses may be expressed as follows, for $u_0 = \kappa\mu,\ \kappa \in \{0,1,2,\ldots\}$:

$$
\begin{aligned}
\psi_P(\kappa\mu) &= \Pr\left\{N\left(\frac{1}{(1+\theta)\lambda}\right) \geq \kappa+1\right\} \\
&\quad + \sum_{j=0}^{\kappa} \Pr\left\{N\left(\frac{1}{(1+\theta)\lambda}\right) = j\right\} \psi_P((\kappa+1-j)\mu) \\
&= \left[1 - \sum_{j=0}^{\kappa} \frac{1}{j!}\left(\frac{1}{1+\theta}\right)^j e^{-\frac{1}{1+\theta}}\right] \\
&\quad + \sum_{j=0}^{\kappa} \frac{1}{j!}\left(\frac{1}{1+\theta}\right)^j e^{-\frac{1}{1+\theta}} \psi_P((\kappa+1-j)\mu) \\
&= 1 - \sum_{j=0}^{\kappa} \frac{1}{j!}\left(\frac{1}{1+\theta}\right)^j e^{-\frac{1}{1+\theta}} \left[1 - \psi_P((\kappa+1-j)\mu)\right]
\end{aligned}
\quad . \tag{A1}
$$

Rearranging equation (A1) yields

$$
\begin{aligned}
& e^{-\frac{1}{1+\theta}}\left[1 - \psi_P((\kappa+1)\mu)\right] \\
&= 1 - \sum_{j=1}^{\kappa} \frac{1}{j!}\left(\frac{1}{1+\theta}\right)^j e^{-\frac{1}{1+\theta}} \left[1 - \psi_P((\kappa+1-j)\mu)\right] - \psi_P(\kappa\mu) \\
& \Leftrightarrow \psi_P((\kappa+1)\mu) \\
&= 1 - \left[1 - \psi_P(\kappa\mu)\right] e^{\frac{1}{1+\theta}} + \sum_{j=1}^{\kappa} \frac{1}{j!}\left(\frac{1}{1+\theta}\right)^j \left[1 - \psi_P((\kappa+1-j)\mu)\right]
\end{aligned}
,
$$

for $\kappa = \{1,2,3,\ldots\}$, or equivalently,

$$
\begin{aligned}
\psi_P(\kappa\mu) &= 1 - \left[1 - \psi_P((\kappa-1)\mu)\right] e^{\frac{1}{1+\theta}} \\
&\quad + \sum_{j=1}^{\kappa-1} \frac{1}{j!}\left(\frac{1}{1+\theta}\right)^j \left[1 - \psi_P((\kappa-j)\mu)\right],
\end{aligned}
$$

for $\kappa = \{2,3,4,\ldots\}$. This proves part (i) for $\kappa \geq 2$. For the case of $\kappa = 0$, it is well-known that

$$
\psi_P(0) = \frac{1}{1+\theta}
$$

for the classical net worth process with linear income and compound Poisson losses (see, e.g., Bowers, et al., p. 415). We then solve the case $\kappa = 1$ by substituting $\psi_P(0) = \frac{1}{1+\theta}$ into the recursive relation (A1).

Part (ii) follows immediately from (i) by taking the appropriate limits. Part (iii) can be demonstrated by considering what happens for fixed u_0 as $\mu \to \infty$. Since the probability of ruin is invariant over simultaneous scale transformations of both u_0 and μ, it follows that the probability of ruin for fixed u_0 as $\mu \to \infty$ must approach the probability of ruin for fixed μ as $u_0 \to 0$.

Proof of Theorem 1 : From Dufresne and Gerber (1991), we know that the ruin probability must be of the form

$$\psi_M(u_0) = C_1 e^{-r_1 u_0} + C_2 e^{-r_2 u_0}, \tag{A2}$$

where r_1 and r_2 are the roots of the equation

$$\frac{\lambda}{(1/\mu) - r} + \frac{\beta}{2} r = \alpha + (1+\theta)\lambda\mu, \tag{A3}$$

and

$$C_1 = \frac{r_1 - (1/\mu)}{1/\mu} \frac{r_2 - r_1}{r_1}, \; C_2 = 1 - C_1. \tag{A4}$$

It is straightforward to show that the roots of the quadratic equation (A3) are given by

$$r = \frac{\alpha + (1+\theta)\lambda\mu}{\beta} + \frac{1}{2\mu} \pm \sqrt{\left[\frac{\alpha + (1+\theta)\lambda\mu}{\beta} + \frac{1}{2\mu}\right]^2 - \frac{2[(\alpha/\lambda) + \theta\mu]}{\beta}},$$

which can be rewritten as

$$r = \frac{\alpha + (1+\theta)\lambda\mu}{\beta} + \frac{1}{2\mu} \pm \sqrt{\begin{aligned}&\left[\frac{\alpha + (1+\theta)\lambda\mu}{\beta} - \frac{1}{2\mu} + \frac{\lambda}{\alpha + (1+\theta)\lambda\mu}\right]^2 \\ &+ \frac{\lambda}{\mu[\alpha + (1+\theta)\lambda\mu]} - \left[\frac{\lambda}{\alpha + (1+\theta)\lambda\mu}\right]^2\end{aligned}}$$

$$\Leftrightarrow r = \frac{\alpha+(1+\theta)\lambda\mu}{\beta} + \frac{1}{2\mu} \pm \sqrt{\left[\frac{\alpha+(1+\theta)\lambda\mu}{\beta} - \frac{1}{2\mu} + \frac{\lambda}{\alpha+(1+\theta)\lambda\mu}\right]^2 + O(h^2)}$$

$$\Leftrightarrow r_1 = \frac{2[\alpha+(1+\theta)\lambda\mu]}{\beta} + \frac{\lambda}{\alpha+(1+\theta)\lambda\mu} + O(h^2),$$

$$r_2 = \frac{1}{\mu} - \frac{\lambda}{\alpha+(1+\theta)\lambda\mu} + O(h^2) \quad .$$

Substituting these roots into the system (A4) yields

$$C_1 = \frac{\frac{2(\alpha+\theta\lambda\mu)}{\beta} + O(h)}{\frac{2[\alpha+(1+\theta)\lambda\mu]}{\beta} + O(h)} = \frac{\alpha+\theta\lambda\mu}{\alpha+(1+\theta)\lambda\mu} + O(h) \quad ,$$

and therefore

$$C_2 = 1 - \left[\frac{\alpha+\theta\lambda\mu}{\alpha+(1+\theta)\lambda\mu} + O(h)\right] = \frac{\lambda\mu}{\alpha+(1+\theta)\lambda\mu} + O(h).$$

Thus, the solution provided by equation (A2) is

$$\begin{aligned} \psi_M(u_0) &= C_1 e^{-r_1 u_0} + C_2 e^{-r_2 u_0} \\ &= \left[\frac{\alpha+\theta\lambda\mu}{\alpha+(1+\theta)\lambda\mu} + O(h)\right] e^{-\left\{\frac{2[\alpha+(1+\theta)\lambda\mu]}{\beta} + \frac{\lambda}{\alpha+(1+\theta)\lambda\mu} + O(h^2)\right\} u_0} \\ &\quad + \left[\frac{\lambda\mu}{\alpha+(1+\theta)\lambda\mu} + O(h)\right] e^{-\left[\frac{1}{\mu} - \frac{\lambda}{\alpha+(1+\theta)\lambda\mu} + O(h^2)\right] u_0} . \end{aligned} \tag{A5}$$

For $u_0 = \kappa\mu + O(h)$, equation (A5) may be rewritten as

$$\psi_M(u_0) = \left[\frac{\alpha+\theta\lambda\mu}{\alpha+(1+\theta)\lambda\mu} + O(h)\right] e^{-O(1/h)} + \left[\frac{\lambda\mu}{\alpha+(1+\theta)\lambda\mu} + O(h)\right] e^{-\left[\frac{\alpha+\theta\lambda\mu}{\alpha+(1+\theta)\lambda\mu} + O(h)\right]\kappa} \quad ,$$

and we obtain the limit in part (i) of the theorem. For fixed u_0, equation (A5) may be rewritten as

$$\psi_M(u_0)=\left[\frac{\alpha+\theta\lambda\mu}{\alpha+(1+\theta)\lambda\mu}+O(h)\right]e^{-\left\{\frac{2[\alpha+(1+\theta)\lambda\mu]}{\beta}+O(h)\right\}u_0}$$
$$+\left[\frac{\lambda\mu}{\alpha+(1+\theta)\lambda\mu}+O(h)\right]e^{-O(h)},$$

which yields the limit in part (ii).

Proof of Theorem 2 : (i) For $u_0 = \kappa\mu + O(h)$, Theorem 1(i) shows that

$$\hat{\psi}_A(u_0)=\lim_{h\to 0}\psi_A(u_0)=\frac{w\pi}{\alpha+(1+\theta)w\pi}e^{-\left[\frac{\alpha+\theta w\pi}{\alpha+(1+\theta)w\pi}\right]\kappa}.$$

Consequently,

$$\frac{d\hat{\psi}_A(u_0)}{dw}=\frac{w\pi}{\alpha+(1+\theta)w\pi}e^{-\left[\frac{\alpha+\theta w\pi}{\alpha+(1+\theta)w\pi}\right]\kappa}$$
$$\times\left\{-\kappa\frac{[\alpha+(1+\theta)w\pi]\theta\pi-(\alpha+\theta w\pi)(1+\theta)\pi}{[\alpha+(1+\theta)w\pi]^2}\right\}$$
$$+\left\{\frac{[\alpha+(1+\theta)w\pi]\pi-w(1+\theta)\pi^2}{[\alpha+(1+\theta)w\pi]^2}\right\}e^{-\left[\frac{\alpha+\theta w\pi}{\alpha+(1+\theta)w\pi}\right]\kappa}$$
$$=\frac{\alpha\pi e^{-\left[\frac{\alpha+\theta w\pi}{\alpha+(1+\theta)w\pi}\right]\kappa}}{[\alpha+(1+\theta)w\pi]^2}\left[\frac{w\pi\kappa}{\alpha+(1+\theta)w\pi}+1\right],$$

which is positive for all $w \geq 0$.

(ii) For fixed u_0, Theorem 1(ii) shows that

$$\hat{\psi}_A(u_0)=\lim_{h\to 0}\psi_A(u_0)$$
$$=\frac{\alpha+w\theta\pi}{\alpha+(1+\theta)w\pi}e^{-\frac{2[\alpha+(1+\theta)w\pi]u_0}{\beta}}+\frac{w\pi}{\alpha+(1+\theta)w\pi}.$$

Consequently,

$$\frac{d\hat{\psi}_A(u_0)}{dw} = \frac{\alpha+\theta w\pi}{\alpha+(1+\theta)w\pi} e^{-\frac{2[\alpha+(1+\theta)w\pi]}{\beta}u_0}\left[-\frac{2(1+\theta)\pi u_0}{\beta}\right]$$

$$+\frac{[\alpha+(1+\theta)w\pi]\theta\pi-(\alpha+\theta w\pi)[(1+\theta)\pi]}{[\alpha+(1+\theta)w\pi]^2} e^{-\frac{2[\alpha+(1+\theta)w\pi]}{\beta}u_0}$$

$$+\frac{[\alpha+(1+\theta)w\pi]\pi-w\pi[(1+\theta)\pi]}{[\alpha+(1+\theta)w\pi]^2}$$

$$=\frac{1}{\alpha+(1+\theta)w\pi}$$

$$\times\left\{-\left[\frac{2(\alpha+\theta w\pi)(1+\theta)\pi u_0}{\beta}+\frac{\alpha}{\alpha+(1+\theta)w\pi}\right]e^{-\frac{2[\alpha+(1+\theta)w\pi]}{\beta}u_0}\right.$$

$$\left.+\frac{\alpha\pi}{\alpha+(1+\theta)w\pi}\right\}$$

It then follows that

$$\left.\frac{d\hat{\psi}_A(u_0)}{dw}\right|_{w=0} = \frac{1}{\alpha}\left\{-\left[\frac{2\alpha(1+\theta)\pi u_0}{\beta}+\pi\right]e^{-\frac{2\alpha}{\beta}u_0}+\pi\right\},$$

which implies

$$\left.\frac{d\hat{\psi}_A(u_0)}{dw}\right|_{w=0} < 0$$

$$\Leftrightarrow \theta > \frac{\left(e^{\frac{2\alpha u_0}{\beta}}-1-\frac{2\alpha u_0}{\beta}\right)}{\frac{2\alpha u_0}{\beta}} = \sum_{i=1}^{\infty}\frac{1}{(i+1)!}\left(\frac{2\alpha u_0}{\beta}\right)^i.$$

Proof of Theorem 3 : for $u_0 = \kappa\mu + O(h)$, Theorem 1(i) shows that

$$\hat{\psi}_S(u_0) = \lim_{h\to 0}\psi_S(u_0) = \frac{w\pi}{(1-w)\alpha+(1+\theta)w\pi} e^{-\left[\frac{(1-w)\alpha+\theta w\pi}{(1-w)\alpha+(1+\theta)w\pi}\right]\kappa}$$

Consequently,

$$\frac{d\hat{\psi}_S(u_0)}{dw} = \frac{w\pi}{(1-w)\alpha+(1+\theta)w\pi} e^{-\left[\frac{(1-w)\alpha+\theta w\pi}{(1-w)\alpha+(1+\theta)w\pi}\right]\kappa}$$

$$\times\left\{-\kappa\frac{[(1-w)\alpha+(1+\theta)w\pi](-\alpha+\theta\pi)-[(1-w)\alpha+\theta w\pi][-\alpha+(1+\theta)\pi]}{[(1-w)\alpha+(1+\theta)w\pi]^2}\right\}$$

$$+\left\{\frac{[(1-w)\alpha+(1+\theta)w\pi]\pi-w\pi[-\alpha+(1+\theta)\pi]}{[(1-w)\alpha+(1+\theta)w\pi]^2}\right\} e^{-\left[\frac{(1-w)\alpha+\theta w\pi}{(1-w)\alpha+(1+\theta)w\pi}\right]\kappa}$$

$$=\frac{\alpha\pi e^{-\left[\frac{(1-w)\alpha+\theta w\pi}{(1-w)\alpha+(1+\theta)w\pi}\right]\kappa}}{[(1-w)\alpha+(1+\theta)w\pi]^2}\left[\frac{w\pi\kappa}{(1-w)\alpha+(1+\theta)w\pi}+1\right],$$

which is positive for all $w \geq 0$.

(i)For fixed u_0, Theorem 1(ii) shows that

$$\hat{\psi}_S(u_0) = \lim_{h\to 0}\psi_S(u_0)$$

$$= \frac{(1-w)\alpha+w\theta\pi}{(1-w)\alpha+(1+\theta)w\pi} e^{-\frac{2[(1-w)\alpha+(1+\theta)w\pi]u_0}{(1-w)^2\beta}} + \frac{w\pi}{(1-w)\alpha+(1+\theta)w\pi}.$$

Consequently,

$$\frac{d\hat{\psi}_S(u_0)}{dw} = \frac{(1-w)\alpha+\theta w\pi}{(1-w)\alpha+(1+\theta)w\pi} e^{-\frac{2[(1-w)\alpha+(1+\theta)w\pi]}{(1-w)^2\beta}u_0}$$

$$\times\left\{-\frac{2u_0}{\beta(1-w)^4}\left\{(1-w)^2[-\alpha+(1+\theta)\pi]+2[(1-w)\alpha+(1+\theta)w\pi](1-w)\right\}\right\}$$

$$+\frac{[(1-w)\alpha+(1+\theta)w\pi](-\alpha+\theta\pi)-[(1-w)\alpha+\theta w\pi][-\alpha+(1+\theta)\pi]}{[(1-w)\alpha+(1+\theta)w\pi]^2}$$

$$\times e^{-\frac{2[(1-w)\alpha+(1+\theta)w\pi]}{(1-w)^2\beta}u_0}$$

$$+\frac{[(1-w)\alpha+(1+\theta)w\pi]\pi-w\pi[-\alpha+(1+\theta)\pi]}{[(1-w)\alpha+(1+\theta)w\pi]^2}$$

$$= \frac{1}{(1-w)\alpha+(1+\theta)w\pi}\left\{-\left\{\frac{2u_0[(1-w)\alpha+(1+w)\pi(1+\theta)][(1-w)\alpha+\theta w\pi]}{\beta(1-w)^3}\right.\right.$$

$$\left.\left. + \frac{\alpha\pi}{(1-w)\alpha+(1+\theta)w\pi}\right\}\times e^{-\frac{2[(1-w)\alpha+(1+\theta)w\pi]}{(1-w)^2\beta}u_0} + \frac{\alpha\pi}{(1-w)\alpha+(1+\theta)w\pi}\right\}.$$

It then follows that

$$\left.\frac{d\hat{\psi}_S(u_0)}{dw}\right|_{w=0} = \frac{1}{\alpha}\left\{-\left\{\frac{2\alpha[\alpha+(1+\theta)\pi]u_0}{\beta}+\pi\right\}e^{-\frac{2\alpha}{\beta}u_0}+\pi\right\},$$

which implies

$$\left.\frac{d\hat{\psi}_S(u_0)}{dw}\right|_{w=0} < 0$$

$$\Leftrightarrow \theta > \frac{\left(e^{\frac{2\alpha u_0}{\beta}}-1-\frac{2\alpha u_0}{\beta}\right)}{\frac{2\alpha u_0}{\beta}} - \frac{\alpha}{\pi} = \sum_{i=1}^{\infty}\frac{1}{(i+1)!}\left(\frac{2\alpha u_0}{\beta}\right)^i - \frac{\alpha}{\pi}.$$

Proof of Theorem 4 : Equation (7) describes a diffusion process satisfying the backward equation

$$\frac{\partial p(u;u_0,t)}{\partial t} = \frac{1}{2}\beta\frac{\partial^2 p(u;u_0,t)}{\partial u_0^2} + \gamma\frac{\partial p(u;u_0,t)}{\partial u_0} - \lambda p(u;u_0,t) + \lambda\int_0^{u_0} p(u;(u_0-x),t)f_X(x)dx \quad \text{(A6)}$$

where $p(u;u_0,t)$ is the probability density of $U_M(t)$ at the point u given that : (1) the process starts from $U_M(0) = u_0$, and (2) the process does not reach the absorbing barrier of zero prior to time t.

Integrating both sides of (A6) over the interval (u, ∞), it can be shown that the survival function, $\zeta_M(t;u_0) = Pr\{U_M(\tau) > 0$ for all $\tau \in (0,t) \mid U_M(0) = u_0\}$, also satisfies the backward equation. Specifically,

$$\frac{\partial \zeta_M(t;u_0)}{\partial t} = \frac{1}{2}\beta\frac{\partial^2 \zeta_M(t;u_0)}{\partial u_0^2} + \gamma\frac{\partial \zeta_M(t;u_0)}{\partial u_0} - \lambda\zeta_M(t;u_0) + \lambda\int_0^{u_0}\zeta_M(t;u_0-x)f_X(x)dx,$$

subject to the boundary conditions $\zeta_M(t;u_0)=0$ and $\zeta_M(t;\infty)=1$. Taking limits as $t\to\infty$, we find that $1-\psi_M(u_0)=\lim_{t\to\infty}\zeta_M(t;u_0)=\zeta_M(u_0)$ must satisfy

$$0=\frac{1}{2}\beta\frac{\partial^2\zeta_M(u_0)}{\partial u_0^2}+\gamma\frac{\partial\zeta_M(u_0)}{\partial u_0}-\lambda\zeta_M(u_0) +\lambda\int_0^{u_0}\zeta_M(u_0-x)f_X(x)dx, \qquad \text{(A7)}$$

subject to the boundary conditions $\zeta_M(u_0)=0$ and $\zeta_M(\infty)=1$.

Since $X(t)$ can take on only one value, its probability density function is given by the Dirac delta function; i.e.,

$$f_X(x)=\delta(x-\mu)=\begin{cases}0 \text{ for } x\neq\mu\\ 1 \text{ for } x=\mu\end{cases},$$

which has the property

$$\int_0^\infty g(x)\delta(x-\mu)dx=g(\mu).$$

Therefore, substituting $f_X(x)$ into equation (A7) yields the ODE (8).

References

Bowers, N. L., H.U. Gerber, J.C. Hickman, D.A. Jones, and C.J. Nesbitt, 1997, *Actuarial Mathematics*, 2nd Edition. Schaumber, IL Society of Actuaries.

Cummins, J. D. (1988), « Risk-Based Premiums for Insurance Guaranty Funds », *The Journal of Finance*, no 43, pp 823-839.

Cummins, J. D., C.M. Lewis, and R.D. Phillips, 1999, « Pricing Excess-of-Loss Reinsurance Contracts Against Catastrophic Loss » in Froot, K.A. ed., *The Financing of Catastrophe Risk*. Chicago, The University of Chicago Press.

Dufresne, F. and H.U. Gerber, 1991, « Risk Theory for the Compound Poisson Process that is Perturbed by Diffusion », *Insurance: Mathematics and Economics,* no 10, pp 51-59.

Froot, K. A., 1999, Introduction, in Froot, K.A. ed., *The Financing of Catastrophe Risk*. Chicago, The University of Chicago Press.

Froot, K. A. and P.G.J. O'Connell, 1999, « The Pricing of U.S. Catastrophe Reinsurance », in Froot, K.A. ed., *The Financing of Catastrophe Risk*. Chicago, The University of Chicago Press.

Gerber, H. U., 1970, « An Extension of the Renewal Equation and Its Application in the Collective Theory of Risk », *Scandinavian Actuarial Journal*, pp 205-210.

Hoerger, T. F., F. Sloan, and M. Hassan, 1990, « Loss Volatility, Bankruptcy, and the Demand for Reinsurance », *The Journal of Risk and Insurance,* no 57, pp 221-245.

Iglehart, D.L., 1969, Diffusion Approximations in Collective Risk Theory. *Journal of Applied Probability*, 6, pp 285-292.

Lewis, C. M. and K.C. Murdock, 1996, « The Role of Government Contracts in Discretionary Reinsurance Markets for Natural Disasters », *The Journal of Risk and Insurance,* 63, 4, pp 567-597.

Merton, R. C., 1976, « Option Pricing When Underlying Stock Returns Are Discontinuous », *Journal of Financial Economics,* no 3, pp 125-144.

Powers, M. R., 1995, « A Theory of Risk, Return, and Solvency », *Insurance : Mathematics and Economics,* 17, 2, pp 101-118.

Powers, M. R., 2002, "Leapfrogging" the Variance : The Financial Management of Extreme-Event Risk, in J. J. Choi and M. R. Powers, eds., *Global Risk Management : Financial, Operational, and Insurance Strategies*, Amsterdam, JAI/Elsevier Science.

Schmidli, H., 1994, « Diffusion Approximations for a Risk Process with the Possibility of Borrowing and Investment », *Communications in Statistics—Stochastic Models,* 10, 2, pp 365-388.

□ Notes

1. Powers (2002) actually considers a somewhat simpler model with random loss amount $X = I \times Y$, where $I \sim$ Bernoulli(p), $Y \sim$ Normal (μ,σ^2), I and Y are statistically independent, and limits are taken as $p \to 0, \mu \to \infty$, $p\mu \to \pi \in (0,\infty)$, and $\frac{\sigma}{\mu} \to 0$.

2. Actually, for the case of constant loss amounts, $\psi_S(u_0)$ is pictured as less than $\frac{1}{1+\theta}$ because the ODE (10), unlike the equation in Theorem 1(ii), is not an asymptotic expression (as $h \to 0$).