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Random Reflections on Science and Calculation

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Random Reflections on Science and Calculation

The purpose of our forthcoming *Introduction to the Philosophy of Nature** will be to explain the subject and principles of what Aristotle calls the science of nature or natural philosophy. The modern reader need hardly be warned that the very first of our difficulties concerns the interpretation of ancient nomenclature. Words and phrases like 'science,' 'subject of a science,' 'principles of a science,' and 'science of nature,' no longer bear the significance which they possessed in ancient times. Indeed it would be difficult to find a single instance of a term like these which has kept its old meaning. Plainly, this is a condition which it would be fatal to overlook.[1]

Many a teacher charged with an elementary course in the Philosophy of Nature — a subject which sometimes goes under the title of what is really only one of its parts, viz. 'Cosmology' — will feel impatient at the solicitude to be shown in the *Introduction* for the scientific climate of our day. Why bother about it? he may protest. We have the mandate to teach a subject, so why not get down to business? But there precisely is the question : can we reasonably get down to it? Is there such a subject? We, on our part, may be already convinced that there is ; but that can scarcely be the point when good teaching demands that we begin from what is known to the listener. Now, the information — even if it be only what he could gather from press headlines — with which a modern pupil is equipped by the time he turns up for an elementary course in any branch of philosophy, is very different from that of the beginner of a mere half-century ago. To ignore that difference will mean to compound confusion. None of the philosophers whom we hold in esteem ever thought he could afford to neglect the opinions of his times. We should be abandoning their ideals, as well as acting unfairly, if we allowed a student to believe that what is meant by 'science' in Aristotle or Aquinas must be roughly the same as what is meant by science today. He will find out eventually that they really have no more in common than the proverbial dog and constellation which go by the same name. It is quite impossible to ignore great contemporary researches, the fruits of which have grown with such cosmic violence. Something

* Prentice-Hall, Englewood Cliffs, N.J.

1. "It is well to remind ourselves sometimes that everything written about ancient philosophy by modern scholars is, to a greater or less extent, vitiated and falsified by the linguistic exchange of currency, and by the underlying shift in the scope and content of concepts" (F. M. CORNFORD, *The Unwritten Philosophy and other Essays*, Cambridge 1950, p.40).

has simply got to be done to show how the old science is related to the new.

I. THE SCIENCE OF NATURE AND THE USE OF WORDS

The first thing to be noted is that in our work all the doctrine will be expressed by means of words, and not by means of symbols. Now, it has lately become obvious that the giant strides in the mathematical study of nature are concomitant with a gradual emancipation from the use of words. Until he is allowed to use symbols that are not names, the mathematical physicist is not sure what he is saying. But notice how this very statement about the use of symbols rather than words uses nothing but words, and it is difficult to see how such statements could be made in any other way. One might of course suggest that our statement be represented by the symbol S ; but the interpretation of the symbol would of necessity carry us back to the statement made in words.

When Sir Arthur Eddington shows so convincingly that the ' exact science of nature ' can get nowhere until it has reduced definitions to measure-numbers, and that these are expressed in terms of mathematical symbols rather than words, he uses words to explain this. Even the terms ' exact,' ' science,' ' symbol ' and ' nature,' he must employ as words, intended to mean something in the way that words do. Indeed, without words he cannot explain how the physicist obtains his measure-numbers and why his concern is only with them. By length, for instance, which is otherwise defined as ' what is extended in one dimension,' he, as a mathematical physicist, means ' when we take a reasonably fair copy of a certain platinum-iridium bar kept in Paris . . . and apply it, once or more, successively or by division, to know the distance between A and B, the result of the operation may be expressed by Lx.' Thus defined, the standard of length can of course have no length, when there is no other standard. Length, then, only comes into the foreground when the measurement is actually made. Weight, in turn, is defined by ' when using a weighing-machine . . .,' and so on for all the basic definitions. Now it should be noticed that the crucial term in these definitions is *when*. If the physicist said ' length is . . .' instead of ' length is when . . .' he would revert to a mode of definition which seeks to tell ' what ' a thing is absolutely, and not merely what a name or symbol is intended to stand for. Having thus defined length, he may assert that " this *is* length," but he can only mean that this understanding of length is the only one with which he will concern himself. In mathematical physics definitions are no more than interpretations of the symbols chosen, or descriptions of how the measure-numbers are obtained. It may be helpful to note that, if this type of definition, in which ' when ' is an essential factor, were the only valid one, the definition of ' man ' would have to be

something like this : " when I tread on something and it produces a series of sounds like ' Where do you think you're going ? ', this *is* man." In other words, all definitions would be interpretations of names or of symbols.

It is also plain that, by his interpretation of the time-symbol t, the mathematical physicist will not intend even a nominal definition of the word ' time,' as this term was and is used without specific reference to the way in which the measure-number is obtained. The same holds for the very expression ' mathematical physics,' meaning a certain type of knowledge about ' nature.' He will never try to define in terms of measure-numbers what the *word* ' nature ' stands for, although it is true that even his kind of definition has something to do with what we call nature. Take, for instance, the following statement made by Einstein : " It is my conviction that pure mathematical construction enables us to discover the concepts and the laws connecting them which give us the key to the understanding of the phenomena of Nature. Experience can of course guide us in our choice of serviceable mathematical concepts ; it cannot possibly be the source from which they are derived ; experience of course remains the sole criterion of the serviceability of mathematical construction for physics, but the truly creative principle resides in mathematics." He makes clear what he means by physics when he adds that by itself such pure mathematical construction " can give us no knowledge whatsoever of the world of experience ; all knowledge about reality begins with experience and terminates in it. Conclusions obtained by purely rational processes are, so far as Reality is concerned, entirely empty." [1] How he would interpret the names ' Nature,' and ' Reality,' we do not know, though he might suggest that to the physicist they are what the measure-numbers refer to in some fashion, and the test of the relevance of rational construction to his purpose. But it is certain that he would not have confined himself to ' Nature is when using such or such a standard of measure . . . etc.,' — even though, in doing so, there would indeed be a certain reference to nature, and to what he already knew ' reality ' to mean.

II. THE SYMBOLIC WORLD OF MATHEMATICAL PHYSICS, AND THE ' SYMBOLICALLY CONSTRUCTED FICTIONS ' OF MATHEMATICAL LOGIC

From the mathematical physicist's standpoint the world is a symbolic one. What Eddington makes clear is that his knowledge of this world can be conveyed only by symbols and involves a generous share of fiction ; it starts from metrical structure and constantly refers

1. *On the Method of Theoretical Physics*, Herbert Spencer Lecture, Oxford, 1933, pp.7, 12.

to no more than metrical structure. He also realizes that whatever the symbols express cannot be all that there is to the world under examination by the one who uses them. Now, once he has made all this clear in words, he goes on using words — and using them with great skill — to bring home his further thoughts on the subject. Hence, to employ either words or symbols is not a matter of choice. According to what we wish to express, now one, now the other, is imposed upon us. We are sometimes led to believe that the use of symbols is merely a way of economizing words. This is not the whole truth. It is essential to realize that the mathematical physicist, as well as the mathematician, does not use symbols instead of names merely for the sake of abbreviating his equations, but because, if expressed in names, the equations could not be solved in the proficient and mechanical way which these require.

As we shall see, the art of calculation simply cannot deal with objects in the sense of what names refer to, like 'man,' 'horse,' or 'nature.' Even what the ancients named 'number' or 'figure' is of no formal interest to calculation itself. "Mathematicians do not study objects," Poincaré said, "but the relations between the objects; it is therefore indifferent to them when the objects are replaced by other objects, so long as the relations do not change. Not the matter, but the form is their concern." [1] And the objects that are of no concern to him are not merely things like horse or apple, but numbers and figures as well.

Now what about objects that neither mathematician nor mathematical physicist is concerned with? What has happened to the numbers, for example, (let it be 'three,' 'four,' or any such number you please) to which we had given names before putting them into an equation? or to the 'time' we named before we manufactured a measure-number by means of the clock? Our manipulation of the symbols may have been so skilful, exact and productive of results that we forgot, or came to think it right to ignore, what those names meant

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1. *La Science et l'hypothèse*, p.32. — To illustrate LORD BERTRAND RUSSELL'S statement that "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true," the French mathematician M. JACQUES HADAMARD chooses the following example: "Having bought 6 metres of cloth at 12 francs a meter, how much does one have to pay? In raising this problem, are we really talking about cloth? Not at all. Instead of asking the price of 6 metres of cloth at 12 francs a meter we could just as well have asked the price of 6 pounds of meat at 12 francs a pound. We might have replaced the meat by copra, and the pupil could have provided the answer without even asking the teacher what copra is. Hence, in raising this problem one does not know what one is talking about; or, to put it otherwise, there is no need to know it. Here, then, in a first, simple instance, we have the notion of mathematical abstraction." In *Encyclopédie française*, section "Mathématique," 1.52-3. — I do not think that M. HADAMARD wishes to explain here all that is intended in LORD RUSSELL'S statement, inasmuch as the numbers in question are themselves only material with regard to more abstract forms which say nothing about numbers.

while we were using them. But can we truly replace what the word 'man' means by referring henceforth only to the swarm of electrical charges which is the mathematical physicist's view of him? A swarm of electrical charges man is, no doubt, but is this 'what it is to be a man'? It is doubtless true also that if the physicist could actually produce such a swarm, he would produce a man ; but why should we call it a man, unless it were like what we have already given that name? [1]

III. WHERE WORDS REMAIN IN USE

The mathematician and the mathematical physicist are only hampered by the use of words while pursuing their type of knowledge but, when they want to convey what their knowledge is about, and especially what it is not about, it seems that they must use them. Even as they assert that they cannot be concerned with things as named, they are using names to make this assertion,[2] although it must be admitted that it is not as mathematicians or mathematical physicists that they make it.

The question we are trying to raise is this : can there be true knowledge about the things of nature as we name them? Can the

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1. In *Human Knowledge* (Allen and Unwin, London 1948), LORD RUSSELL rightly observes : " All nominal definitions, if pushed back far enough, must lead ultimately to terms having only ostensive definitions, and in the case of an empirical science the empirical terms must depend upon terms of which the ostensive definition is given in perception. The Astronomer's sun, for instance, is very different from what we see, but it must have a definition derived from the ostensive definition of the word 'sun' which we learnt in childhood. Thus an empirical interpretation of a set of axioms, when complete, must always involve the use of terms which have an ostensive definition derived from sensible experience. It will not, of course, contain *only* such terms, for there will always be logical terms ; but it is the presence of terms derived from experience that makes an interpretation empirical.

" The question of interpretation has been unduly neglected. So long as we remain in the region of mathematical formulae, everything appears precise, but when we seek to interpret them it turns out that the precision is partly illusory. Until this matter has been cleared up, we cannot tell with any exactitude what any given science is asserting " (p.258). Reprinted by permission of the publishers. — This must also apply to each and every word used in philosophy and in logic, but not to the symbols of logical calculus. The latter " symbolize *directly* the thing talked about " (JAMES R. NEWMAN, *The World of Mathematics*, p.1852). They need no interpretation, and the operations upon them are entirely mechanical ; whereas names and verbs signify the things named, not absolutely, but only as we know them. The operations of calculation, being purely mechanical, can be entrusted to machines ; no perception of relation is essential to computing.

2. " Indeed, the first difficulty the man in the street encounters when he is taught to think mathematically is that he must learn to look things much more squarely in the face ; his belief in words must be shattered ; he must learn to think more concretely. Only then will he be able to carry out the second step, the step of abstraction where intuitive ideas are replaced by purely symbolic construction " (HERMANN WEYL, " The Mathematical Way of thinking," reprinted from *Science*, Nov. 15, 1940, in *The World of Mathematics*, edited by JAMES R. NEWMAN ; Simon and Schuster, New York, 1956, p.1834). Reprinted by permission of the Editors of *Science*.

things named be the source of further knowledge about themselves, to be further expressed by names? Again, can the things that names may refer to be defined, and used in proofs, in a way which deserves to be called scientific? Or must the term 'science' be restricted to the art of calculation and its application? Before they were defined by measure-numbers, what did men mean by 'change,' 'movement,' 'infinity,' and 'time?' Has their old meaning now become mere deception?

It has been suggested that the only reason for continuing to use words is that they are necessary to communication in the order of behaviour — that language is essentially practical. No court of law, for example, would excuse manslaughter as being no more than a disturbance produced in a particular swarm of electric charges by another swarm reasonably like the first. So we are allowed to go on believing that Mr. Smith is *there* in some fashion or other which is perhaps far from clear, and that, after all, he still has rights and obligations, just as we do. But it seems that, so soon as we forget about the practical order, — about how we should behave and treat our neighbour, and all such affairs described in words — and apply ourselves to scientific investigation, things like man and his doings are to be irretrievably abandoned. If the thing (even 'thing' may sound distressingly unscientific) we call 'man' does persist, it is only as what turns up for breakfast, or is summoned to pay taxes, or allowed to sleep, and in some cases even allowed to study physics.

It is no doubt significant that words are used to tell us these things, and that these things would not be told unless, in using words, our thoughts seemed directed to what was recognized as their meaning. Nor is it without significance that practical life should force their use upon us. There is no denying that many of the words which for centuries remained basic in philosophy, like 'matter,' 'form,' 'action,' originally referred to the practical order, to the order of making and doing, and not to things of nature; and 'time' may well have meant originally something we never have enough of. Surely such facts are worth looking into, however little of the scientific spirit there may now seem to be in curiosity about them.

IV. IF ALL DEFINITIONS WERE TO BE INTERPRETATIONS OF NAMES OR OF SYMBOLS

If it must be assumed that there can be no true knowledge of things as we name them, but only of that which can be expressed by the symbols of calculation, then what is so stated in words can hardly be true. Let us put it still another way. If, as John Stuart Mill said, "All definitions are of names, and of names only," in the sense that the things named cannot be defined in themselves, however tentative-

ly, and that we cannot know what they are, but only what the name is that signifies them ; and if there cannot be a science of the names themselves, inasmuch as they signify no more than by convention, it is clear that there can be no science of anything to the extent that it is named.

What Mill believes of names applies literally to the symbols of the art of calculation, whether used in mathematics or in physics. To define a symbol, as we have explained already, is simply to interpret the symbol by explaining how it is to be taken, not by stating *what* the thing is to which it refers. For instance, when asked to define the number two, the art of calculation will not try to tell us what two is. What two is never enters into the operation of calculating ; in that operation, two is only a term with a function similar to that which it fills in an equation like $2 + x = 5$. Whether two, here, is actually ' one two ' or ' two ones ' will make no difference to the art. The only unity 2 possesses in such an equation is the unity of a symbol ; and whatever sort of unity 2 may enjoy apart from that assigned to it as an operational symbol is quite irrelevant to a definition derived from its operational use alone. Lord Russell puts it this way :

We naturally think that the class of couples (for example) is something different from the number 2. But there is no doubt about the class of couples : it is indubitable and not difficult to define, whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive. Accordingly we set up the following definition : —

The number of a class is the class of all those classes that are similar to it.

Thus the number of a couple will be the class of all couples. In fact, the class of all couples will *be* the number 2, according to our definition.[1]

It is admittedly difficult to see how any other way of being two could be relevant to the equation $2 + x = 5$. In this context, therefore, Aristotle's definition of number as ' a plurality measurable by the unit' must appear awkward, and is certainly useless. But Aristotle was trying to convey what number is, not what an operational symbol may stand for.[2]

Definitions of the symbol type appear in connection with geometry as well. Hermann Weyl had this to say in illustration of what he meant by ' creative definitions ' :

Thus, in plane geometry, the concept of a circle is introduced with the help of the ternary point relation of congruence, $OA = OB$, which appears in

1. *Introduction to Mathematical Philosophy*, London, 1930, p.18. Reprinted by permission of the publishers, Allen and Unwin.

2. *Metaphysics*, X, chap.6, 1057 a.

the axioms, as follows, "A point O and a different point A determine a circle, the 'circle about O through A'; that a point P lies on this circle means that $OA = OP$." For the mathematician it is irrelevant what circles are. It is of importance only to know in what manner a circle may be given (namely by O and A) and what is meant by saying that a point P lies on the circle thus given. Only in statements of this latter form or in statements explicitly defined on their basis does the concept of a circle appear.[1]

Especially deserving of attention is the precise statement that "For the mathematician it is irrelevant what circles are." Further on, Hermann Weyl puts down his understanding — most mathematicians now share his view — of what is meant by the 'concept' of number :

If one wants to speak, all the same, of numbers as concepts or ideal objects, one must at any rate refrain from giving them independent existence ; their being exhausts itself in the functional role which they play and their relations of more or less. (They certainly are not concepts in the sense of Aristotle's theory of abstraction.) [2]

Returning now to the mode of definition in mathematical physics, we have Eddington's incontrovertible statement about what a definable weight is : "Never mind what two tons *refers* to ; what *is* it? How has it actually entered in so definite a way into our experience? Two tons *is* the reading of the pointer when the elephant was placed on a weighing-machine." [3] There is never any attempt to reveal what weight is apart from this particular mode of defining, viz. by describing how the physicist obtains his measure-number.

V. WHAT IS IMPLIED BY THE STATEMENT THAT SCIENCE IS NO LONGER CONCERNED WITH 'OBJECTS'

We have been told that the mathematician is not concerned with objects, that he cannot get very far with a number, like two, of which Lord Russell says that it "is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down." [4] To the geometer, as we have seen, it is also irrelevant 'what circles are.' We must be aware of the implication of this fact with regard to what was previously called mathematical science, and which had to do with

1. HERMANN WEYL, *Philosophy of Mathematics and Natural Science*, Princeton, 1949, p.8. Reprinted by permission of the publishers, Princeton University Press.

2. *Op. cit.*, p.36. Reprinted by permission of the publishers, Princeton University Press.

3. *The Nature of the Physical World*, Cambridge, 1931, p.251.

4. RUSSELL'S statement is amply supported by the history of philosophy from earliest times. On the other hand, his own definition is not intended to solve the problem of what number is, absolutely ; it states 'what number is to the calculator.' Note, too, that the Aristotelian would not call number a metaphysical entity, except by extrinsic denomination, meaning that its definition is achieved in first philosophy.

quantity, quantity being either number, the subject of arithmetic, or continuous quantity, the subject of geometry. According to Aristotle, these subjects are to be defined in metaphysics, whereas the mathematician takes them for granted. Today, the mathematician does not assume them, he is perfectly content with symbolic construction or creative definition. But it is important to note that these constructions do not so much as attempt to account for the definable natures which Aristotle had in mind. The latter are simply left out, because, it is said, we can never feel sure that they exist or that we have tracked them down. Those who persist in the attempt to track them down would appear to be pursuing will-o-the-wisps.[1]

We may perhaps make clear what has happened by comparing what Poincaré declared to be the concern of the mathematician — viz. the form, and not the object that he also calls the matter — with what the Greeks called the matter and the form of a number. Aristotle distinguishes a matter and a form that constitute a number intrinsically, and which are related to each other as potency to act. The matter of a number is the units that compose it in the order of material cause, like the pieces of wood that make up a table, or, better still, like the limbs that make up the body of a man. By the form of the number, he meant the particular kind of unity and order which is exhibited by the adding of a unit to a unit, of a unit to the number so obtained, and so on for all the integers.[2] This addition does not fabricate the number, but merely brings to mind new kinds of number which, though they are not conceived as existing in reality in the way that Socrates

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1. According to ARISTOTLE, mathematical science does not establish its own subject, nor does it justify its principles ; it assumes them. " With what sort of things must the mathematician be supposed to deal ? Not surely with the things in this world ; for none of these is the sort of things which the mathematical sciences investigate . . . [ST. THOMAS explains : ' For in these sensible things, there are no lines and circles, such as those which the mathematical sciences investigate.'] Nor again does the science of which we are now in search [i.e., first philosophy] treat of the subjects of mathematics [in the way in which the mathematician deals with them], for none of these has separate existence . . . In general one might raise the question, to what kind of science it belongs to discuss the difficulties concerning what the mathematical sciences are about. Neither to physics — because the whole inquiry of the physicist is about the things that have in themselves a principle of movement and rest — nor yet to the science which seeks demonstration and science [from and about such a subject] ; for *this* is just the subject which *it* investigates. It remains then that it is the philosophy which we have set before ourselves [i.e., first philosophy] that treats of those subjects. » *Metaph.*, XI, chap.1, 1059 b 5 ; cf. ST. THOMAS, *ibid.*, lect.1, (edit. Cathala) n.2161ff. — To ARISTOTLE's mind, a mathematician's attempt to establish the subject of his science or to justify its principles would end in complete frustration. Which reminds one of the opinion of the late HERMANN WEYL : " In spite, or because, of our deepened critical insight we are today less sure than at any previous time of the ultimate foundations on which mathematics rests " (" The Mathematical Way of Thinking," in *The World of Mathematics*, p.1849). See also JOHN VON NEUMANN, *The Mathematician*, *ibid.*, pp.2053-2063.

2. *Metaph.*, VIII, chap.3, 1044a.

does, nevertheless are thought of as endowed with certain properties which are true even when not actually being considered by the mind. Number, thus understood, is defined as 'a plurality measurable by the unit' — this being the principle of number. Now, any proper measure must be one in kind with the measured, meaning, here, that 'to be in number,' the constituent units must be of the same nature. The particular kind of unity that is proper to any given number depends upon the homogeneity of its components. Otherwise we have no more than "a sort of heap." [1] The number two, then, is not the same as two mere units.

Still, even when objects are not of the same kind, we can nevertheless count them, like the objects in this room — persons, desks, chairs, coughs, absences, the relations of reason that we have in our mind, and even those which we ought to have but do not. There must therefore be a number that applies to the heterogeneous elements of a heap, or to a mere aggregate, a number which we use simply to express how many objects are there. This type of number arises in the act of sheer counting. It is the number characteristic of that art of calculation which was called *logismos* or *logistikè*. Whatever unity such a number may have is provided by the operations of addition, multiplication, subtraction and division. Hence, its unity is in no way based on the nature of the things which are added, multiplied, subtracted or divided ; besides, whether these have a nature or not is equally irrelevant to the operations upon the symbols. It is this number which has been defined as the class of all those classes which are similar to it. Thus the number 2 is the class of all couples, no matter what their kind or the kind of their elements. Nor do the couples or their units have to be couples or units in any positive sense, for if number is defined by the operation, whatever the operation may be applied to will by that very fact be such a number, like zero, or a fraction, or an irrational number. Number, thus understood, is not an object in the sense in which the number two that is 'one two' is an object. It is a convenient fiction which our mind has produced. Though it be a fiction, it is nonetheless effective, as can be seen from the fact that by means of it we can count things regardless of what they are ; and this is of course because '*what* the things are' is of no account to the calculator. The indifference of this number to the nature of the numbered is equalled only by the indifference of the elements of a heap to their neighbours in the heap. Whether they belong together or not, the mind can put them together for a purpose alien to their nature or to their lack of it.

The science of arithmetic, as Aristotle and Euclid understood it, is about the numbers that are *per se* one ; unlike *logismos*, it does not abstract from what the things are to which it is applied. Like the

1. *Ibid.*

subject of any science, the numbers must be one *per se*. What Whitehead says about arithmetic in the following passage will therefore hold good only of the art of calculation which the science of mathematics, in the ancient sense, employs :

> Now, the first noticeable fact about arithmetic is that it applies to everything, to tastes and to sounds, to apples and to angels, to the ideas of the mind and to the bones of the body. The nature of the things is perfectly indifferent, of all things it is true that two and two make four. Thus we write down as the leading characteristic of mathematics that it deals with properties and ideas which are applicable to things just because they are things, and apart from any particular feelings, or emotions, or sensations, in any way connected with them. This is what is meant by calling mathematics an abstract science.[1]

Perhaps we ought to make it explicit that the nature of things is indifferent to the point where all that Whitehead mentions might be gathered under a single number.

VI. THE EXPRESSION 'MATHEMATICAL SCIENCE' NOW HAS A NEW MEANING

Arithmetic, as understood in the above passage, can have nothing whatsoever to do with the subject of the science which the ancients called by the same name. In fact, most moderns would say that what the ancients had in mind was not a science at all. This is what Lord Russell implies when he says its subject would have to be something "about which we can never feel sure that it exists or that we have tracked it down." On the other hand, there is no doubt about the class of couples : anything, thing or no, can belong to it, if it is a couple, and no matter what it is a couple of. Thus mathematics, as understood today, has put aside everything that may in any way be called into question. To get hold of what is left we do not even have to determine whether anything corresponds to the fictions, nor even whether these are fictions, with an existence only in the mind. To save their value, even 'logical,' as in 'logical fictions,' does not have to be tied down to what is in or of the mind.[2] It is enough that 'logical'

1. *Introduction to Mathematics*, London, 1931, p.9.

2. "When we have decided that classes cannot be things of the same sort as their members, that they cannot be just heaps or aggregates, and also that they cannot be identified with propositional functions, it becomes very difficult to see what they can be, if they are to be more than symbolic fictions. And if we can find any way of dealing with them as symbolic fictions, we increase the logical security of our position, since we avoid the need of assuming that there are classes without being compelled to make the opposite assumption that there are no classes. We merely abstain from both assumptions" (B. RUSSELL, *Introduction to Mathematical Philosophy*, p.184). Reprinted by permission of the Editors, Allen and Unwin.

should refer to *logismos*. Assuredly, it cannot refer to logic in the Aristotelian sense of this term. The latter is concerned with a particular type of relations of reason, which are called second intentions.[1]

A further point is worth noting here. The art of calculation does not take into account whether a number is a group of actually divided elements, or whether it is a one that is divisible yet not divided. Whatever is to the right of the symbol of equality is essentially the same as what is to the left of it. Thus $1 + 1 = 2$ is *exactly* the same as $1 + 1 = 1 + 1$. Hence, whether 2 stands for what may be one two, or for two ones of any kind, is completely indifferent.[2] The number for which it stands may be actually one or actually many, it makes no difference here. Such is the case with all the basic laws of the art of calculation. We may ignore, then, whether a number is " an aggregate of units, as is said by some [e.g. Thales, who is said to have defined number as a bundle of units] ; for two is either not one, or the unit is not present in it in complete actuality." [3] The same will be true for magnitude : whether the line is actually divided, or only potentially so, is irrelevant to the art of calculation when applied to it. Moreover, whether a line contains an infinity of points in potency or in act, is indifferent : of the infinite no more is required than that we should be able to define it operationally. The distinction between act and potency is beside the question. Infinite classes can be easily defined in this manner, and whether there is indeed an infinite class, in the way that there is a number *per se* one is a matter irrelevant to that which this art defines (in the above-mentioned manner) and to which it

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1. *Vd.* SHEILA O'FLYNN, " The First Meaning of ' Rational ' Process . . .," in *Laval théologique et philosophique*, 1954, Vol.X, n.2.

2. This has been clearly exhibited by COURANT and ROBBINS, in *What is Mathematics?* Oxford University Press, 1951, chap.1. Also HERMANN WEYL, " The Mathematical Way of Thinking," *loc. cit.*, pp.1832-1849.

3. *Metaph.*, VII, chap.13. Here is the context : " A substance cannot consist of substances present in it in complete actuality ; for things that are thus two in act [i. e. each having a complete and distinct actuality of its own], are never one in act, whereas if they are two only in potency, they can be one [in act], like things that are double, as two halves potentially ; for the complete actualization of the halves divides them one from the other ; therefore if the substance is one, it will not consist of substances present in it and present in this way, which Democritus describes rightly ; he says one thing cannot be made out of two nor two out of one ; for he identifies substance with his indivisible magnitudes. It is clear therefore that the same will hold good of number, if number is no more than an aggregate of units, as is said by some ; for two is either not one, or the unit is not present in it in act " (1039 a 1-15). We use the Oxford translation. — RICHARD VON MISES, in *A Study of Human Understanding* (Harvard 1951), quotes a significant passage from GOETHE which shows how little deceived the poet was by what now goes under the name of mathematics : " Mathematics has the completely false reputation of yielding infallible conclusions. Its infallibility is nothing but identity. Two times two is not four, but it is just two times two, and that is what we call four for short. But four is nothing new at all. And thus it goes on and on in its conclusions, except that in the higher formulas the identity fades out of sight."

applies. To the art of calculation, such questions must always be pointless and obstructive.

VII. THE 'MATHEMATICS' THAT ABSTRACTS FROM THE DISTINCTION BETWEEN *PER SE* AND *PER ACCIDENS*.

All this implies that *logismos* side-steps the distinction between what is *per se* and what is *per accidens*, either as to being or as to unity. That the mind can transcend this division is plain from the fact that nothing prevents it from stringing together the following : 'bald-headed pale barn-building flute-playing thrice-married ill-tempered barber,' where the connections are all plainly *per accidens*, (otherwise it would be impossible to be one of those things without being the other too). We cannot name what it is to be such a particular accidental ensemble — although it may be true of 'Oscar' [1] — but it is the easiest thing in the world to let a symbol stand for it. In terms of the calculus of classes, anything which is all those things together belongs to the class that is the logical product of the classes 'bald-headed,' 'pale,' 'barn-building,' etc., and this product may be represented by the single arbitrary sign Ψ.

This kind of abstraction may lead to certain paradoxes which we are faced with only because we are still using names about the elements concerned. Take for instance the principle that the whole is greater than any of its parts. It has been argued that this principle does not always apply, and therefore is not universal. Consider, for example, the series of whole numbers compared to the series of even numbers, as Lord Russell presents them :

$$\begin{array}{cccccc} 1, & 2, & 3, & 4, & 5, & 6\ldots \\ 2, & 4, & 6, & 8, & 10, & 12\ldots \end{array}$$

" There is one entry in the lower row for every one in the top row ; therefore the number of terms in the two rows must be the same, although the lower row consists of only half the terms in the top row." [2] He denies that it is a contradiction, " it is only an oddity." He must conclude nonetheless that it contradicts the statement that every dimensional or numerical whole is greater than its part.

Why does the comparison of these series appear to deny the generality of this principle? Because the word 'part' is used equivocally. The comparison here made neglects the distinction between a number and an individual instance of that number ; between the *series* of integers and a particular instance of the series ; between

1. *Vd.* St. Thomas, *De Ente et Essentia*, c.3.

2. *A History of Western Philosophy*, Simon and Schuster, New York, 1945, p.829. Reprinted by permission of the publishers.

species and individual, universal and particular. There is only one number two, but there are as many individual twos as we wish ; there is only one series of whole numbers, but there are an infinity of particular instances of the series. Such distinctions are of course irrelevant to the mechanics of calculation.

When Lord Russell says that " The number of even numbers must be the same as the number of all whole numbers," by ' number ' he cannot mean " the series of whole numbers," which he has just mentioned in the same paragraph. For if we remove the even numbers from that series, we are left with no more than the series of odd numbers, and the comparison is between the series of odd and that of even, and they are equal ; one is not part of the other ; they are severally parts of the series of whole numbers which is greater than either.[1]

The matter is somewhat different when we take a particular instance (A) of the series of integers and compare it to a particular instance (B) of the series of even numbers, or to one (C) of the series of odds. In this case, B is not part of A. It is another individual series illustrating the series of even numbers, which is part of the series of integers. Now, if A is taken as an illustration of the whole series of integers, and B as an instance of part of that series, then B will compare to A, which contains both odd and even, as a part, so that for every particular even number in B there will be an even and an odd in A. But if we set A and B in one to one correspondence, as in the two rows :

(A) 2, 3, 4, 5, 6, . . .
(B) 2, 4, 6, 8, 10, . . .

then B is not compared to A as a part to a whole, but as one whole to another of the same number. The principle that every dimen-

1. " That which is infinite in every way can be but one. Hence the PHILOSOPHER says (*De Coelo*, I, cc.2-3) that, since bodies have dimensions in every part, there cannot be several infinite bodies. Yet if anything were infinite in one way only, nothing would hinder the existence of several such infinite things ; as if we were to suppose several lines of infinite length drawn on a surface of finite breadth. Hence, because infinitude is not a substance, but is accidental to things that are said to be infinite, as the PHILOSOPHER says (*Phys.*, III, chap.5) ; as the infinite is multiplied by different subjects, so, too, a property of the infinite must be multiplied, in such a way that it belongs to each of them according to that particular subject. Now it is a property of the infinite that nothing is greater than it. Hence, if we take one infinite line, there is nothing greater in it than the infinite ; so, too, if we take any one of other infinite lines, it is plain that each has infinite parts. Therefore of necessity in this particular line there is nothing greater than all these infinite parts ; yet in another or a third line there will be more infinite parts besides these. We observe this in numbers also, for the species of even numbers are infinite, and likewise the species of odd numbers are infinite ; yet there are more even and odd numbers than even. And thus it must be said that nothing is greater than the simply and in every way infinite ; but than the infinite which is limited in some respect, nothing is greater in that order ; yet we may suppose something greater outside that order " (ST. THOMAS, *IIIa Pars*, q.10, a.3, ad 3).

sional or numerical whole is greater than its part remains unchallenged.

The oddity of the argumentation which can lead a man to a denial of the generality of this principle may become still clearer if it be noted that one might just as readily run a single number, odd or even, into an infinite series by taking it over and over ; for any number of the series of integers can be taken as many times as there are numbers in the series. Consider the two rows :

$$2,\ 3,\ 4,\ 5,\ 6, \ldots$$
$$2,\ 2,\ 2,\ 2,\ 2, \ldots$$

The second row, made up of repeated twos, would now be equal to the series of whole numbers and, by the same argument which was employed above, it would follow that a single number of the series is equal to the entire series.[1]

What has brought about this oddity? There is nothing strange in the fact that of any single integer there can be as many individual instances as there are integers. The oddity arises when the whole-part principle is interpreted to mean, for some unasserted reason, that the infinite series of whole numbers ought to be greater than the infinite series of individual instances of any single integer. Is it odd that between two and four there is only one whole number although an infinity of threes is possible? Only to the thinker who has failed to notice that an individual three is not a part with regard to the number three in the sense in which the latter is part of the series of integers. When he assumes that these widely different orders can be placed on the same footing (as when a number is held to be no more than a collection or a collection of collections, a logical fiction, a symbolic construction ; and which might stand if the distinction between universal and singular could be forever ignored) he will encounter ' oddity ' only because he has not succeeded in banishing from his mind number in the sense of that " metaphysical entity about which we can never feel sure that it exists or that we have tracked it down." When orders so widely different are taken as comparable, paradoxical results are inevitable.

What is there to prevent us, for example, from pointing out a man, with two legs, as a walking contradiction : there is only one man, and he has two legs, parts of him, so that the parts are greater than the whole? To achieve this oddity, all that is necessary is to forget that, in making our comparison, we first cease to consider the legs as parts, and then still treat them as parts. To sum up, once a writer resorts to what was called symbolic construction or creative definition,

1. Proceeding in this fashion, one might point out that the infinite series of twos is even greater than that of the integers, seeing that there is a first positive integer, but no first two.

he should realize that he may no longer use names; or should bear in mind that they are linguistic devices sure to cause confusion in the measure that they continue to evoke what can no longer be intended.

A study of Lord Russell's writings will show that all the amusing paradoxes contained in them find their explanation in the fact that their author is constantly betrayed into treating that which is one *per se* and that which is only *per accidens* one as if they were on the same footing. This is good enough for the calculator, but it leads the philosopher inevitably into comedy. 'Mr. Smith,' for him, is a mere bundle of events. An example like this one would not appear funny, it would not seem hopelessly incongruous, but for the clash between the *per se* unity which we cannot help but keep in mind, and the mere incidental whole suggested by the term bundle. It recalls the comic cartoons of the elephant eating up the jam and following his trunk so eagerly that, in the last sketch, he has swallowed himself entire ; for it is plain that he is no longer there. The cartoonist, if he is expected to be funny, must be permitted to treat things in this way : he can then go Lord Russell one better, and make the part devour the whole before our eyes. To forbid such a procedure to humourists would be to make impossible the Cheshire cat and most of the other delightful characters in *Alice in Wonderland.*

The reader should be aware by now that we are attempting to take notice of views which are both striking and widely current. If we have singled out one of the most distinguished proponents of these opinions, it is because he is a writer whom we neither wish to ignore, nor could ignore if we wished. Bertrand Russell has earned the admiration of men like Sir Arthur Eddington whose philosophy of physical science we continue to defend. But Lord Russell has at times strayed a long way from the field in which he is at home, as one can see more especially in *A History of Western Philosophy*, a survey which brought all of western philosophy toppling down, and a good deal of his own with it. Although his sweeping negations are not accepted in many quarters, still their author has made undeniably valuable contributions to what he himself terms the 'scientific outlook.' Where this scientific outlook begins and where it ends, however, he does not make quite clear. If it is a mode of approach which must be extended over all fields of thought, (with the possible exception of ethics), as he appears to believe, then there can be no doubt that most of the teachings of earlier philosophers must be discarded. The *History*, in fact, assures us that all philosophers, save those of the modern school which Russell claims for his own, held arbitrary doctrines and defended them with reasons which, as Lord Russell states them, are downright silly.

But it must be noted that, even if this scientific outlook is not properly extensible over those domains which used to be called scientific in a quite different sense, this does not tell against its genuine

intrinsic value. What he wants too much of is still a good thing. And there may also be dialectical profit in trying to determine what would happen if he had his way, if the new outlook were allowed all the scope he could ask for. Such a venture into doubtful matters may land us in situations so curious as to give the impression that irony is being abused, or that unbalance is contagious. Let the reader entertain no doubt about our own position in the matter. Our aim is to disturb contentment where we deem it to be illusory, by making clear, in dialectical fashion, what happens when the Russell alternative is pursued to the limit. But let it be repeated that, even though his view can be proved untenable as a general one, it can still be shown to be valid in some fields and to some degree — precisely in what fields and to what degree is not the question here.

VIII. WHETHER TO ASK WHAT A THING IS HAS NOW BECOME IRRELEVANT

As for the two modes of defining distinguished earlier, we hold them to be both valid, and fail to see any contradiction in doing so. No objection can be made to defining man as a rational animal and then, for quite different purposes, to interpreting the name as a symbol by ' when I tread on something . . . etc.' There is in fact a domain where definitions by interpretation of name or symbol are the only ones to promote profitable research. The second type of definition of man can actually lead to " detailed and precise knowledge of normal and pathological mental processes in a desired direction and thus cure mental ailments."[1] We find little in Aristotle's *De Anima* to advance knowledge along those lines, although what Aristotle teaches may do something to convince one that the subject is deserving of relief. But the mere fact that the first type of definition can provoke endless discussion, and the latter little or none, should not make us prefer one to the other. If we gave up ' rational animal ' as hopeless and chose the narrower definition, there is no doubt that nowadays we would receive credit for being broader of mind. The attempt to get the best out of both alternatives will certainly appear to some a sheer waste of time. But surely we may be allowed to try. If our right to freedom of thought is to mean anything, then it must embrace the freedom to risk being considerably wrong and even of holding positions no longer tenable. To turn a natural right to such sorry employment may make one unworthy of the company of the right-minded ; but the risk might be worth taking, even if the only result were to help protect freedom of thought for those who can make better use of it.

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1. Dr Franz Alexander, Introduction to *What Man has Made of Man*, by Mortimer Adler, Chicago, 1937, p.xi.

It is an historical fact that, so long as the study of the physical world made essential use of names, little was achieved to further knowledge of the kind now called physics. Where the Greek philosophers sought to know *what* the things of nature are, we appear to have renounced that type of inquiry for the simple reason that it does not lead to the kind of knowledge about nature actually obtained by another type of method, whose possibilities have only begun to reveal themselves.[1] Is there however any good reason why the former mode of investigation should be abandoned altogether and everywhere? Is it always beside the point to be interested in objects and to ask what things are? The physicist, from the very outset, defines movement by the way he measures it, and that is what movement *is* to him. But does this mean that it can never be anything but irrelevant to ask what movement is, apart from this operational way of defining it? Today there is fairly general agreement that such questions are of their nature futile.

The entire treatise of Aristotle, which has come down to us under the name *Physics*, deals with a few definitions and a relatively small number of demonstrations, most of which must appear outlandish when we look at them in the light of what is now called physics. His intention seems clear : he wanted to provide in this work a general introduction to the study of nature, while later treatises would show how this general science must branch out into particular sciences whose denominations we have in some instances retained. In Book I of this work he investigates the principles of the subject of natural science in its widest acceptation. In Book II, having exposed some meanings of the term ' nature,' he determines what kind of knowledge we are after, what are the causes, or definitions, from which demonstration can be obtained in this field. It is here that he raises the problem of how the natural scientist and the mathematician differ when talking somehow about the same subjects ; finally he shows the difference between necessity in mathematics and the kind of necessity found in nature and in the science of nature. Book III starts with what movement is and, after defining it as something admittedly obscure, deals next with the problem of how movement is related to infinity and of what infinity is. Book IV is about place and time. The discussion is somewhat uneven inasmuch as assumptions are made which, though not essential to his arguments, do rest on theories (expounded in later treatises) which eventually went the way of the opinion that the earth was the hub of the universe and that the stars are where we see them. An example is his identification of time, which he first defines correctly and independently, with the movement of the ' outer sphere.' These assumptions, then, are no more than

1. Cf. F. M. CORNFORD, " Greek Natural Philosophy and Modern Science," *op. cit.*, pp.81ff.

incidental to the definitions arrived at. Book V is about the division of movement into its kinds, viz. movement according to quantity, quality and place. He then presents a few notions such as ' to be in contact,' ' between,' 'next to,' 'contiguous,' ' continuous,' thus leading towards the discussion of movement according to its quantitative parts in Book VI. In this book he first defines ' continuum,' ' indivisibles ' and ' infinitely divisible ; ' and makes a first approach to Zeno's paradoxes, which are left unresolved until Book VIII. Both the exposition and solution of Zeno's problem differ widely from those which would be made in terms of sheer calculation. Books VII and VIII culminate in demonstration of a first mover unmoved. The whole of the discussion makes no sense in terms of mathematical physics, nor was it ever intended to have such a meaning, of course, or to convince by the same means.

Nearly everyone holds that whatever interest the *Physics* may now possess can be no more than historical. This we interpret as a challenge, not so much to the particular doctrines it contains but, what is far more important, to the meaning and validity of the kind of questions its author assumes the human mind should be facing. The questions are of the kind which still lead philosophers to the most contradictory positions, but whether that fact by itself provides sufficient reason for refusing even to consider them is, to my mind, debatable. It is interesting to note that sharply conflicting opinions arise most easily when the philosophers themselves fail to appreciate where the true difficulties are and assume that they are solved before they have even given them proper investigation. Descartes' thinking was an example of this, when he insisted that movement was one of the clearest things known,[1] an assertion unquestionably true of what he had in mind, but which reveals that he had no adequate understanding of the question.

Whatever the case may be as to relevance, we leave it to the reader to judge the extent to which we may still be allowed to ask, and in words, just what it is that the study of nature is about ; whether it is possible to define movement in the sense of ' what ' it is, and not merely to interpret the word by pointing out some instance of it, like " Mr. Smith moved from street A to street B," and then to allow the physicist to define it in his own way ; whether it is possible to tell what time is, or only to tell the time, and so on.

IX. A DIFFICULTY CONCERNING THE VERY NAME ' SCIENCE '

Now, the first difficulty we meet is the meaning of the very term ' science ' in the expression ' science of nature ' or ' natural science.' In this matter, Aristotle himself appears to be of little help since in

1. *Regulae ad directionem ingenii,* XII.

the *Posteriora Analytica* (I, i-ii), where he is illustrating what he means by 'to possess unqualified scientific knowledge of a thing,' he refers to the demonstrations of mathematics. If we follow him and choose an example from geometry, like the first proposition in Euclid : 'On a given finite straight line to construct an equilateral triangle,' we find ourselves in an awkward situation, in view of what is commonly held today about Euclid's mode of demonstration. Here, for instance, is what Lord Russell has to say on the subject — and, bearing in mind the kind of rigour he demands, I cannot see that one could disagree with him :

The rigid methods employed by modern geometers have deposed Euclid from his pinnacle of correctness. It was thought, until recent times, that, as Sir Henry Savile remarked in 1621, there were only two blemishes in Euclid, the theory of parallels and the theory of proportion. It is now known that these are almost the only points in which Euclid is free from blemish. Countless errors are involved in his first eight propositions. That is to say, not only is it doubtful whether his axioms are true, which is a comparatively trivial matter, but it is certain that his propositions do not follow from the axioms which he enunciates. A vastly greater number of axioms, which Euclid unconsciously employs, are required for the proof of his propositions. Even in the first proposition of all, where he constructs an equilateral triangle on a given base, he uses two circles which are assumed to intersect. But no explicit axiom assures us that they do so, and in some kinds of spaces they do not always intersect. It is quite doubtful whether our space belongs to one of these kinds or not. Thus Euclid fails entirely to prove his point in the very first proposition. As he is certainly not an easy author, and is terribly longwinded, he has no longer any but an historical interest. Under these circumstances, it is nothing less than a scandal that he should still be taught to boys in England. A book should have either intelligibility or correctness ; to combine the two is impossible, but to lack both is to be unworthy of such a place as Euclid has occupied in education.

The most remarkable result of modern methods in mathematics is the importance of symbolic logic and of rigid formalism.[1]

Thus, on the one hand we are faced with Aristotle's manifest conviction that the proposition about the construction of an equilateral triangle is an instance of true demonstration, and on the other hand by a modern mathematician's assurance that this proof is so sadly lacking in rigour that it can never serve as a specimen of what a philosopher means by demonstration. The problem becomes all the more puzzling when we learn that Aristotle and his followers saw in mathematics the archetype of what science means to us : for in it, they held, that which is most knowable in itself is also that which is most knowable to us — adding that this could never be the case in metaphysics.

1. *Mysticism and Logic*, London, 1932, pp.94-95. Reprinted by permission of the publishers, Allen and Unwin.

X. THE IDEAL OF RIGOUR IN MATHEMATICAL LOGIC

To proceed first in a negative way, let us bear in mind that what Aristotle intended by ' science ' and ' mathematics ' is not at all what we usually mean by these words today. Our present Queen of the Sciences, which is called mathematics, mathematical logic, logical mathematics, logistics, and frequently just logic, has an ideal of rigour which, in the ancient mind, would be verified rather of the operations of the calculator in arithmetic. In order to appreciate how thoroughgoing is the modern standard of rigour, we need only consider that the entire operational structure of mathematics or logic ought to be formalized in such a way that

> one is able to see the structures of configurations of certain " strings " (or sequences) of " meaningless " signs, how they hang together, are syntactically combined, nest in one another and so on.
>
> A page covered with the " meaningless " marks of this calculus speaks for itself, as does a mosaic, an abstract design, a geometric diagram ; . . .[1]

What does " calculation " mean ? The word comes from *calculus*, which originally meant " pebble. " Long before the introduction of symbols, pebbles were used in counting, as our chips are used in playing poker. Hence, to calculate meant to make clear ' how many ' units there were in a collection, by comparing it with a collection of pebbles more easily managed. Now, when two or more classes are equal, we say that their number is the ' same number. ' To establish that two or more classes have the same number is to achieve exactness or rigour.[2]

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1. Ernest Nagel and James R. Newman, " Goedel's Proof ", in *The World of Mathematics*, p.1675. Reprinted by permission of the Editors, Simon and Schuster, New York.

2. A difficulty might arise here in connection with the notions of equality, similitude, and sameness or identity. " Equal " is said of objects that are " one in quantity ; " " similar, " when they are " one in quality ; " " same," in the sense of " identical, " when they are " one in substance, " as in : " The man next door, and the one you saw at Mindy's, are the same man." Now, we said that the number of equal classes is the same number; for example, if the number of a group of pebbles is ten, and equal to the number of letters in the name " Washington, " the number of both classes is identical. But this does not mean that they are the same ten ; one group are pebbles, and the other letters. Aristotle explains this in *Physics*, IV, chap.14, 224 a 2 : " It is said rightly, too, that the number of the sheep and of the dogs is the same *number* if the two numbers are equal, but not the same *decad* or the same *ten* ; just as the equilateral and the scalene are not the same triangle, yet they are the same *figure*, because they are both triangles. For things are called the same so-and-so if they do not differ by a differentia of that thing, but not if they do ; e.g. triangle differs from triangle by a differentia of triangle, therefore they are different triangles ; but they do not differ by a differentia of figure, but are in one and the same division of it. For a figure of one kind is a circle and a figure of another kind a triangle, and a triangle of one kind is equilateral and a triangle of another kind scalene. They are the same figure, then, and that, triangle, but not the same triangle. Therefore the number of two groups also is the same number (for their number does not differ by a differentia of number), but it

How can we show, in the light of a simple example, what symbolic construction means? In counting with the fingers of one hand to find the cardinal number of the letters in the word " five, " " five fingers " will *signify* the number of letters. But if, instead of referring to fingers, we put down the symbol 5, the symbol can be used in counting, just as if the symbol itself were the cardinal number of the class.

The cardinal number of the class C is thus seen to be the *symbol* representing the set of all classes that can be put into one-to-one correspondence with C. For example, the number five is simply the name, or symbol, attached to the set of all classes, each of which can be put into one-to-one correspondence with the fingers of one hand.[1]

Why the arbitrary marks used in calculation are meant to " symbolize *directly* the thing talked about " [2] can be seen best in the case of large numbers. If, in performing the operations, we had to abstract from these symbols insofar as they are purely operational, and attend piecemeal to all the elements in the class, as in counting directly with pebbles, even 5×25 would be relatively involved ; let alone 10^{10}, which, as a symbol, is perfectly clear, while no one can visualize such a number any more than an infinite one. This means that in calculating we do not have to interpret the symbols in the operation itself ; which is another way of saying that the operation is purely mechanical. If we had to keep in mind their meaning, as we ought to do when

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is not the same decad ; for the things of which it is asserted differ ; one group are dogs, and the other horses." (Oxford transl.) However, such distinctions and problems are irrelevant to the symbolic construction we are about to consider.

1. EDWARD KASNER and JAMES NEWMAN, *Mathematics and the Imagination*, Simon and Schuster, New York, 1949, p.31. Reprinted by permission of the Editors.

2. JAMES NEWMAN, *The World of Mathematics*, p.1852. — The particular case of geometry and its implication has been very well put by R. COURANT and H. ROBBINS in *What is Mathematics ?*. " From a purely formal point of view, we may start with a line made up only of rational points and then *define* an irrational point as just a *symbol for a certain sequence of nested rational intervals*. An irrational point is completely described by a sequence of nested rational intervals with lengths tending to zero. Hence our fundamental postulate really amounts to a definition. To make this definition after having been led to a sequence of nested rational intervals by an intuitive feeling that the irrational point " exists," is to throw away the intuitive crutch with which our reasoning proceeded and to realize that all the *mathematical properties* of irrational points may be expressed as properties of nested sequences of rational intervals.

" We have here a typical instance of the philosophical position described in the introduction to this book ; to discard the naive "realistic" approach that regards a mathematical object as a " thing in itself " of which we humbly investigate the properties, and instead to realize that the only relevant existence of mathematical objects lies in their mathematical properties and in the relations by which they are interconnected. These relations and properties exhaust the possible aspects under which an object can enter the realm of mathematical activity. We give up the mathematical " thing in itself " as physics gave up the unobservable ether. This is the meaning of the " intrinsic " definition of an irrational number as a nested sequence of rational intervals " (p.69). Reprinted by permission of the publishers, Oxford University Press, New York.

using words, we could get nowhere.[1] The interpretation of the symbols must remain quite extrinsic to the actual operations upon them ; we must prescind from symbols as signs, divorcing them altogether from the order of representation, and commit ourselves to do nothing that a machine could not do. In the process, the operations themselves must, as it were, be kept 'outside the mind' and thus, no less than the symbols, drained of any meaning whatsoever. The rules of operation are just as mechanical as the rules we build into a machine.[2] There was a time when the operation of sorting potatoes according to size was done by a selective judgment and by hand. Now the 'selection' is performed by a machine, and far more efficiently. The difficulty of conceiving such utter detachment in sheer computation is inversely proportional to the ease with which the operations can be carried out. And when the arbitrary marks are called 'abstract symbols,' the abstraction implied must not be referred to what is and goes on in a mechanical computer, but to the knower who may interpret them. The meaningless symbols are the very opposite of abstraction ; they are "out there" in the same way in which the stuff that the marks are made of is there in the machine. Otherwise, machines could not be made to calculate. "For this reason, it has been said that 'in calculation the pen sometimes seems to be more intelligent than the user.'"[3]

It should be noted at this point that mathematics nowadays is held to be exactly this game with meaningless symbols played according to fixed rules ; and that, as Poincaré said, it is no more necessary for the mathematician than it is for these machines to know what he is doing. Like the symbols, the operations themselves are meaningless, until the non-mathematician interprets them.

What strikes us first of all in the new mathematics is its purely formal character. "Imagine," says Hilbert, "three kinds of *things*, which we will call points, straight lines, and planes ; let us agree that a straight line shall be determined by two points, and that, instead of saying that this straight line is determined by these two points, we may say that it passes through

1. "David Hilbert has in our day pursued the axiomatic method to its bitter end where all mathematical propositions, including the axioms, are turned into formulas and the game of deduction proceeds from the axioms by rules which take no account of the meaning of the formulas. The mathematical game is played in silence, without words, like a game of chess. Only the rules have to be explained and communicated in words, and of course any arguing about the possiblilities of the game, for instance about its consistency, goes on in the medium of words and appeals to evidence" (Hermann Weyl, "The Mathematical Way of Thinking," in *The World of Mathematics*, p.1848. Reprinted by permission of the Editors of *Science*.

2. This may help to understand what some philosophers imply when they say that the use of words is inseparable from emotional involvement, as indeed it is in rhetoric, poetry, sophistry, and the practical sciences of behaviour.

3. M. Cohen and E. Nagel, *An Introduction to Logic and Scientific Method* p.120. Reprinted by permission of the publishers, Harcourt, Brace and Company, New York.

these two points, or that these two points are situated on the straight line." What these *things* are, not only do we not know, but we must not seek to know. It is unnecessary, and any one who had never seen either a point or a straight line or a plane could do geometry just as well as we can. In order that the words *pass through* or the words *be situated on* should not call up any image in our minds, the former is merely regarded as the synonym of *be determined*, and the latter of *determine*.

Thus it will be readily understood that, in order to demonstrate a theorem, it is not necessary or even useful to know what it means. We might replace geometry by the *reasoning piano* imagined by Stanley Jevons ; or, if we prefer, we might imagine a machine where we would put in axioms at one end and take out theorems at the other, like that legendary machine in Chicago where pigs go in alive and come out transformed into hams and sausages. It is no more necessary for the mathematician than it is for these machines to know what he is doing.[1]

XI. A RIGOUR THAT IS ACHIEVED APART FROM MENTAL OPERATION

It may seem queer, but the fact is — and it is vital to notice it — that once the symbolic system is set up, every operation proper to the human mind will be excluded from the kind of logic practised by means of the system. Machines can perform all such operations, and sometimes only machines can perform them. In the course of the operation upon symbols, as in passing from x and y to z, the symbols themselves become irrelevant as signs or representations. This is plain from the fact that such symbols may be fed to a machine which will turn out the correct solution of a computation too prolix or involved for the human brain. To produce the correct result, the machine does not have to 'know' that the symbol may stand for something that is not a symbol. Hence, with regard to these operations " man's rationality marks only a difference in degree from other animals, and fundamentally, no difference at all from the machine. For modern computers are essentially logical machines : they are designed to confront propositions and to draw from them their logical conclusions." [2]

To those who believe that computers are therefore endowed with mind just as man is [3] — implying that mind is no more than what is

1. Henri Poincaré, *Science and Method*, trans. F. Maitland, Dover Publications, Inc., New York 1952, p.147.

2. A. Kaplan, " Sociology Learns the Language of Mathematics," reproduced in *The World of Mathematics*, p.1308. Reprinted by permission of the Editors of *Commentary*, New York. In a noteworthy chapter of Mr. James R. Newman's *What is Science* (Simon and Schuster, New York 1955), Mr. Jacob Bronowski observes that " There is nothing recondite in these machines. Their steps are logical, and they are possible because deductive logic can be formalized and therefore mechanized " (p.401).

3. *Vd* A. M. Turing, " Can a Machine Think ?, " in *The World of Mathematics*, pp.2099-2123.

found in the machine while it is computing — it is sometimes pointed out that thought is still needed to interpret the symbols and the result of the machine's operations upon them. But both this assumption and the protest against it have the effect of obscuring the real point. The champions of the thinking-machines are not justified in concluding that the human mind can be dispensed with, so to speak, but they are doing us a service in pointing out that here is an operation which involves no mental activity. It follows that when the human mind does carry it out, it must do so without the aid of any of those operations which were held to be characteristic of the human mind, like apprehension, judging or reasoning, unless we have already identified these with what goes on in a computer. But it is this identification which is implied whenever mind is defined by the sort of operation so efficiently performed by the machine. And it is quite true that something is arrived at by the machine without our thinking about it, somewhat in the way thoughts are contained in a book that no one is now reading — except that the machine writes the book by itself, as it were, since it produces new combinations of symbols which no human eye or mind need ever so much as examine. Something *we* would have had to do has been done by *it*. And this is what is termed a logical operation in the modern sense — if the quotation from Dr. Kaplan is truly representative.[1] Whether the operation goes on in a mind or not is quite irrelevant. It is there — whatever ' there ' may mean — involving symbols, which we can interpret if we wish, but the interpretation of them as relevant to this or that material never was the business of the modern logician, and there is no reason to expect it from the machine.

Let us face the situation squarely. The language used about computers has put us in a predicament, and no amount of ' beautiful poems the machines cannot write ' is going to get us out of it. Suppose the machines start falling in love with one another, as predicted ; surpass Sophocles and Shakespeare ; reproduce themselves, and even take up all our *Lebensraum* ; where do we go *then* ? They may resent having been called ' no more than machines.' Yet none of the literature on the subject shows any cause for distinguishing between machine and non-machine, between natural and artificial ; while ' Nature,' with or without the capital, is surely no more than the appearance of a name — a sign that does not signify. What this literature does convey is that the use of diverse names, and of single names with different meanings, reflects no more than a perversity which somewhere along the line got itself built into our own machinery. Quite seriously, the machines, too, have been spoken of as capable of

1. It appears likely that department stores will eventually have on sale an electronic helmet, a kind of ' logical cap ' to stimulate the brain and set off the kind of operations now identified with mathematics.

perversion ; though compared with man the machine is favoured, being endowed with an innocence which man, if ever he did possess it, has now assuredly lost. In fact, as history in general reveals it, and the history of philosophy in particular, the most striking feature of man — a trait approaching somewhat the 'specific difference' no longer granted him — is the dimension of his perversity. This attitude appears to stand out most clearly when, as an eminent logician states it in a paper entitled *Can A Machine Think ?* :

> We like to believe that Man is in some subtle way superior to the rest of creation. It is best if he can be shown to be *necessarily* superior, for then there is no danger of him losing his commanding position. The popularity of the theological argument is clearly connected with this feeling. It is likely to be quite strong in intellectual people, since they value the power of thinking more highly than others, and are more inclined to base their belief in the superiority of Man on this power.
>
> I do not think that this argument is sufficiently substantial to require refutation. Consolation would be more appropriate : perhaps this should be sought in the transmigration of souls.[1]

This leaves our machines with the problem why some of them, not only think, but crave that to think be a function proper to them, elevating them above other machines, and non machines. It is all very puzzling. Perhaps the solution is to stop thinking altogether, at least in the old sense of the word. Still, it is curious that, in the face of so much stupidity, wrongheadedness, and sophistry the ancient mind could still maintain that rationality was the prerogative of man.

Apropos of words, the ancients made some distinctions which may not be entirely irrelevant to the language employed about computers ; and they might even have been able to explain why in talking about computers we use terms and expressions as we do. Our machines are being 'fed,' they 'count,' 'remember,' 'learn,' 'understand,' 'confront propositions,' 'draw conclusions,' 'reason,' etc. Now there is no doubt that the 'intellectually most courageous' of the writers on thinking machines intend the words they use to mean exactly what they do when referred to human activity. Aristotle, on the other hand, believes that 'to confer' means something proper to reason, and that reason is proper to man, of all animals. This must of course seem strange to our computer-men. For one might easily point out that the first and least disputable meaning of 'to confer' is 'to bring together,' or 'collect ;' as a cat can bring together her kittens, and even the wind can gather the clouds. 'To reason' implies a conferring ; a going from one thing to another, and coming to a stop or conclusion. Water and mirrors have been confronting things down

1. A. M. Turing, "Can A Machine Think ?," reprinted in *The World of Mathematics*, pp.2029-2123. The quotation is from p.2108. Reprinted by permission of Mrs. E. Sara Turing.

through the ages, not to mention how they sometimes operate amazing transformations. There is nothing easier than to use these words without reference to later meanings.[1] The Latin for ' soul,' *anima*, first means ' air, a current of air, a breeze, a wind ; ' so that, on this meaning of the word, air-conditioners will be more animated than we are, and will have power to animate *us* as well. A mere glance at Aristotle's *De Anima* will show that the word *psychè* has many meanings, as widely different as man from shoe. How he justifies new impositions of the same name could be learned only by careful reading.

That the language about computers must reduce to some primitive meanings, which do not refer to anything that sets ' rational ' apart

1. " A fertile and interesting source of change in the meaning of words arises when their application is broadened because of a *metaphorical extension* of their meaning. Thus ' governor ' originally meant a steersman on a boat, ' spirit ' meant breath ; a bend in a pipe is called an ' elbow,' the corresponding parts of a pipe-fitting are called ' male ' and ' female,' and so on " (M. COHEN and E. NAGEL, *op. cit.*, p.119). I do not believe the authors imply that to extend the meaning of a word is to produce a metaphor. At any rate, *we* distinguish between using a metaphor, and converting a word into an analogous term. In the case of metaphor, the *meaning* of the word is not changed ; in ' lion-hearted,' as applied to a man, the original meaning is retained exclusively, and the comparison involved owes its force of expression to the retention of that single meaning. But the analogous term such as ' healthy ' has many meanings, according as it is used to signify the quality of an animal, where health is referred to its proper subject ; to signify a cause of health, such as ' healthy medicine ; ' or a sign of health, as in ' healthy urine.' The extension of the word ' healthy,' to mean more than what is found in the animal, implies a new imposition, with dependence upon, yet comprising, its first. The same would hold for the name ' light,' meaning first of all the light which allows us to see with our eyes ; and then is extended as in ' the light of new evidence.' The latter meaning depends upon the first, while the word now has different, though related, meanings. Similarly, there is something proportionally common to the steersman of a boat and the one who directs a state ; to the timber (the original meaning of ' matter ') we use in building, and the terms of a syllogism, or the letters of a word. Still, it is not always plain whether a word is being used as a metaphor, or as an analogous term. For instance, if ' governor ' is applied to the one who directs a state or province, without a new imposition, it is used as a metaphor ; but if we impose it to mean ' whoever directs or steers in any order,' it becomes either a generic term, or an analogous one. The point we wish to make here is that if extended meaning implied that the word is being used as a metaphor, then the proper sense of any word should be identified either with ' *that whence* the word was taken to signify,' as ' understand ' from ' under ' and ' stand ; ' or with ' *that which* the word was originally intended to mean,' as ' matter ' meant timber. Hence, if a man were said to ' see ' that 10^{10} is a large number, the verb ' see ' would be used as a metaphor. No words such as ' understand,' ' confer,' ' conclude,' and so on, could properly signify anything characteristic of man. This is what Mr. NEWMAN may be understood to imply when he reports that " ARISTOTLE was of the opinion that man is a rational animal because he can count. This may not seem to us a very impressive argument. Arithmetic is easier than it was in ancient times ; the number system has been improved and better methods of calculation have been invented. We use machines which are far more proficient at arithmetic than even the cleverest human computer. It is not surprising, therefore, that arithmetic has lost caste. BERTRAND RUSSELL points out that ' though many philosophers continue to tell us what fine fellows we are, it is no longer on account of our arithmetical skill that they praise us ' " (*The World of Mathematics*, p.488). Reprinted by permission of the Editors, Simon and Schuster, New York.

from 'irrational,' can be seen in John von Neumann's penetrating chapter on *The General and Logical Theory of Automata.*[1] What the language refers to is made clear and concrete, raising problems not less simple than the difference between a 'writing pen' and a 'writing man.' He remains, as he should, on a level where the question of essential differences simply does not arise. Here and there the vocabulary is anthropomorphic, but it is plainly just that. On the other hand, the writers who insist upon assimilation, to the point of identity, of thought as it is in man with what goes on in the machine, are wre tling with shadows. 'Life,' 'matter,' and the like, mean things they read about in text-books, or upon which they themselves have experimented. Now, no modern biology text known to me offers any convincing criteria of life ; the criteria, as they are presented, are mere hypotheses. We should realize that, such being the case, 'non-living' is just as much a hypothesis as life itself. The same holds for 'life' and 'death,' 'matter' and 'spirit,' 'thought' and 'process.' Death is a mere hypothesis depending upon the hypothesis of life. There is no 'scientific' evidence for either, and the problem is raised only when we refer to that which we had thought, and named accordingly, for reasons which are not scientific in the modern sense of this term. Of a broken-down house we do not say that it is dead, except by metaphor. Why should we say it of Mr. Smith? And how can we talk about spirit as 'immaterial,' when there is no strictly scientific evidence for matter?[2] The difference between thought and mechanical process must submit to the same reduction : the distinction will be only in name. When Descartes thought he had an intuition of the very essence of his soul, and of God, these words, as he used them, would eventually allow us 'to build a rational process into a machine,' and make of the Godhead a mental disease.

XII. THE WORD 'LOGIC' HAS RECEIVED A NEW MEANING, AND WHAT IT REFERS TO PRODUCES RESULTS

This may be what is meant by saying that modern logic has attained a rigour and detachment hitherto unknown. Lord Russell, for instance, like other contemporary logicians of repute, has denounced the Aristotelian doctrines of logic as "wholly false, with the exception of the formal theory of the syllogism, which is unimportant."[3] This apparently severe judgment is actually too mild, I think. For the doctrine contained in the *Prior Analytics* is, from the viewpoint of

1. In *The World of Mathematics*, pp.2070-2098.

2. "The notion [of matter] was invented by Aristotle of Stagira in the early part of the fourth century B. C." A. A. LUCE, *Sense without Matter or Direct Perception*, Nelson, Toronto, 1954, p.3.

3. *A History of Western Philosophy*, p.202.

modern logic, far too misleading to be merely unimportant. Its symbols, in effect, viz. A, B, and C for major, middle and minor terms are restricted to a very particular kind of relation of reason arrived at in a way that is completely unintelligible according to the standards of modern logic. Moreover, the very meaning of ' syllogism as to form ' depends, in Aristotle, upon the value of the treatises that precede it in the order of learning. For the purposes of modern logic, Aristotle's doctrines are as aimless as the search for an elephant for the earth to rest upon.

The super-mathematical theory of groups is one of the most abstract branches of the game with symbols. Still, it possesses more than abstract validity, since in the quantum theory, it provides the only means of accounting for what goes on in the atom.[1] One hardly sees how traditional logic, or even mathematical science as Euclid conceived of it, could do such a thing. However, it is significant that in a chapter of *The World of Mathematics*, entitled : *The Supreme Art of Abstraction : Group Theory*, Mr. James R. Newman observes that : " It should be emphasized that while both the elements and operations of a group may theoretically be undefined, if the group is to be useful in science they must in some way correspond to elements and operations of observable experience. Otherwise manipulating the group amounts to nothing more than a game, and a pretty vague and arid game at that, suitable only for the most withdrawn lunatics." [2]

What becomes of mathematics when reduced to a game with symbols played according to fixed rules ? Mr. E. T. Bell, in a paragraph entitled *The Queen of Queens' Slaves*, observes, of the use of calculating machines, that mathematics " has enslaved a few of these infernal things to do some of her more repulsive drudgery. What I shall say about these marvelous aids to the feeble human intelligence will be little indeed, for two reasons : I have always hated machinery, and the only machine I ever understood was a wheelbarrow, and that but imperfectly." [3] Now the crucial question is, where does the ' repulsive drudgery ' begin and where does it end ? If mathematics, or logic, is to be identified with the symbol-game played according to fixed rules, and if the machines can play the game, it appears that the only thing left to the mathematician is the choice of symbols and the setting up of rules ; the choice being so much a matter of whim that it can hardly be left to calculators as rigorous as the machines.

Surely, this cannot be what is meant by mathematics today. Yet it is difficult to see what else it can mean. No doubt operations which may be entrusted to the computers, when they take place in the

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1. EDDINGTON, *New Pathways in Science*, chap.13.

2. P.1535.

3. *Mathematics, Queen and Servant of Science*, Toronto, Copyright 1951, p.247. Reprinted by permission of the publishers, McGraw-Hill Book Company, Incorporated.

head of a mathematician, are somewhat better off, or more at home ; but this can only be inasmuch as they are now associated with elements foreign to the formal structure that is somehow contained and whirling about in the machine. What can the mind claim for its own in this game? Perhaps it is those definitions, discussed on earlier pages, which are constructed in terms of the kind of operation a computer can carry out. If this be so, the mind's contribution is certainly a useful one when applied to the world of experience, but it is hardly science, and assuredly not pure science, if this be defined as the pursuit of knowledge for its own sake, or knowledge possessed for no other reason than to possess it. Yet what else can the mind boast of which is not also found in the machine? Perhaps the delight of setting up problems that can be handled by the computer? But we should then be obliged to ask for what reason man should take pleasure in this kind of work? What is there about it to bring him delight, without referring it to knowledge of another kind? The old Aristotelian logic, at any rate, never offered itself as desirable for its own sake, but only as a kind of discipline, and a painful and weary one, needed as a preparation for the mind's real work. Nor does it look possible for the new logic, when it is called science in its purest form, to provide the joy of seizing truth, since it is " the subject in which we never know what we are talking about, nor whether what we are saying is true." Our special delight then would consist in our not being concerned with truth.

St. Thomas observes that logic and mathematics, besides being sciences, are likewise called arts, " because, in them, there is not only knowledge, but also a work, one that comes directly from reason itself (*opus aliquod, quod est immediate ipsius rationis*) ; such as making a construction, a syllogism or a phrase (*oratio*) ; numbering, measuring, forming melodies and computing the course of the stars."[1] Modern mathematicians have isolated the 'work' and identified it with logic and mathematics. Yet even the ' work ' must be narrowed down to something that is brought about in a strictly mechanical fashion, excluding the respect in which it is *immediate ipsius rationis*. It is therefore difficult to see why the modern emphasis should be on art as creative reason, since reason is eventually identified with its work, namely, the machine.

And so we see to what extent the ideal, in our time, is indeed a civilization of work. It is worthy of note that what there is of art in logic and mathematics is no longer conceived of as ' liberal ;' the stress is now on ' mechanical.' The dignity of man is no longer to be sought in his ability to act according to virtue, and to acquire science and wisdom. The accent is on the worker, at every level of activity. And the activity itself is finally drained of reason as proper to man,

1. *In Boethium de Trinitate*, q.5, a.1, ad 3.

and man therefore is deprived of his own nature, which can no longer exist except as an object of opprobrium.

XIII. WHAT HAPPENS WHEN THE RIGOUR OF THE NEW LOGIC IS REQUIRED OF SCIENCE

If Lord Russell's absolute rigour is to be the canon of science, and if science without that strictness is strictly nonsense, then of course, we shall have none but that which proceeds by way of " symbolically constructed fictions " or " creative definitions," the game with symbols played according to fixed rules. This sacrifices all of geometry in the ancient sense, together with all of arithmetic as concerned with what numbers are apart from operations upon them. Modern logic and mathematics, then, leave us with no more than an elaborate development of what Plato and Aristotle called *logismos* or *logistikè*, viz. the art of calculation used by the mathematician when he demonstrates. For they distinguished between the operations of calculation and the activity of demonstration, even when this involves calculation, as shown by the following example from Euclid's *Elements* (IX, 24) :

If from an even number an even number be subtracted, the remainder will be even.
For from the even number AB let the even number BC, be subtracted :

I say that the remainder CA is even. For, since AB is even [i. e. ' divisible into two equal parts '] it has a half part.
For the same reason BC also has a half part ; so that the remainder [CA also has a half part, and] AC is therefore even. Q. E. D.

Now this demonstration comprises a calculation, namely the subtraction AB – BC = CA. This operation is not the demonstration, although the operation of demonstration depends upon the calculation. The symbols AB and BC stand for terms subject to calculation. But the middle term in this proof, viz. the implied definition of even number (' divisible into two equal parts '), could hardly be symbolized as such, nor is this kind of definition subject to calculation. (An instance of the definitum may be so, but the instance itself could never be the middle term). Now, modern mathematics would apparently retain only what can be symbolized, and execute only the operation upon the symbols.

The ancient mathematician assumed that there *are* even numbers — existence here meaning no more than that we can form true propositions about them, as that they are divisible into two equal parts — and that the construction of their series is no more than a means of

discovering them. Similarly for the basic assumptions of geometry : The ' point ' and the various kinds of continua are assumed to ' exist,' in the sense just noted, and whatever was constructed by their means to be no more than an instrument for the discovery of abstract (in a very special sense of abstract) things and certain of their necessary properties. For instance, in the demonstration of the equilateral triangle, the demonstration is not the construction, nor does it even bear upon the figure *qua* constructed. What the demonstration is actually about is those entities which the modern mathematician considers a nuisance to be done away with, things involving ' naive concepts ' (concepts or notions in the Aristotelian sense), such as ' one,' ' number,' ' point,' ' line,' and whatever can be established by construction with them in demonstration — the basic entities being supplied by some kind of intuition, like *seeing* the intersection of the circles.

Plainly, if these concepts are without validity, no course is left but that of creative definition. Having taken this course, we can indeed move along freely and rigorously and achieve valid results unobtainable by demonstration ; since demonstration neither possesses, nor was ever thought to possess, the rigour attributable to calculation. However, once we have done away with definition in the sense of stating what a thing is, like ' a plane surface bounded by a single line which at every point is equidistant from the point within called its center,' we have also done way with what is the middle term in demonstration, and therefore with mathematical demonstration in the ancient sense. Lord Russell is clearly aware of the effect of this emancipation upon *logismos*, the art previously considered to be no more than the handmaid of mathematical science. The result is that mathematics is concerned merely with what he calls ' logical fictions.' [1]

Now there can be no objection to fictions, logical or otherwise, especially when they can produce results, as in mathematical physics, and in literature. In fact, a geometry of logical fictions (logical in the operational sense of the word) has proved far more useful than that developed as a science acquired by syllogistic demonstration. The older geometry used both logic and the art of calculation, but it was not about logical entities, nor did it have anything to do with the realities of nature. It was concerned with quantity, which can be abstracted in a fashion all its own, and requires no verification in sense-experience. Its propositions were held to be true of the abstract entities *qua* abstracted. No judgment about either physical or metaphysical reality was implied.[2] At the same time, the very fact that it does actually lead to a greater control of nature should prove

1. The current terms ' logic ' and ' logical ' should be referred, of course, not to what ARISTOTLE meant by them, but to what he called *logismos*.

2. ST. THOMAS, *In Boethium de Trinitate*, q.5, a.3, c. ; *In II Physic.*, lect.3.

that we cannot underrate the value of emancipated logismos or mathematical logic. But it is also important to realize that it is held to be about logical fictions and operations upon them, and even that indifference to truth is declared essential to it — this being one of the reasons why the physicist can be helped by this new method to approach the truth about natural things, whereas euclidean geometry, which aimed at the truth about abstract magnitude, cannot render him the same service.[1]

XIV. WHETHER THE ACCURACY OF A(A) IS TO BE DEMANDED IN ALL CASES

Aristotle was always ready to try to see how people come to misunderstand a subject as well as how they come to understand it. There is a passage in the *Metaphysics* (II, ii) which is of great help in exposing the roots of the general difficulty we have been trying to dispose of before our venture into ancient modes of thought. Here is the chapter we have in mind :

The effect which lectures produce on a hearer depends on his habits ; for we demand the language we are accustomed to, and that which is different from this seems not in keeping but somewhat unintelligible and foreign because of its unwontedness. For it is the customary that is intelligible. The force of habit is shown by the laws, in which the legendary and childish elements prevail over our knowledge about them, owing to habit. Thus some people do not listen to a speaker unless he speaks mathematically, others unless he gives instances, while others expect him to cite a poet as witness. And some want to have everything done accurately, while others are annoyed by accuracy, either because they cannot follow the connexion of thought or because they regard it as pettifoggery. For accuracy has something of this character, so that as in trade so in argument some people think it mean. Hence one must be already trained to know how to take each sort of argument, since it is absurd to seek at the same time knowledge and the way of attaining knowledge ; and it is not easy to get even one of the two.

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1. The Euclidean type of geometry falls from its pinnacle as a science in ARISTOTLE's sense, only if taken as a natural science. But was it ever intended to be the geometry of nature ? Certainly not in ARISTOTLE ; nor is there any evidence for such a misunderstanding in EUCLID. The science of geometry is one thing ; its applicability to nature quite another. (Cf. *Physics*, II, chap.2.) The latter could never have been more than an hypothesis. VON NEUMANN's observation is very much to the point : " The prime reason, why, of all EUCLID's postulates, the fifth was questioned, was clearly the unempirical character of the concept of the entire infinite plane which intervenes there, and there only. The idea that in at least one significant sense — and in spite of all mathematico-logical analyses — the decision for or against EUCLID may have to be empirical, was certainly present in the mind of the greatest mathematician, GAUSS. And after BOLYAI, LOBATSCHEFSKI, RIEMANN, and KLEIN had obtained *more abstracto*, what we today consider the formal resolution of the original controversy, empirics — or rather physics — nevertheless, had the final say " (" The Mathematician," in *The World of Mathematics*, p.2055). Reprinted by permission of the Editors, Simon and Schuster, New York.

The minute accuracy of mathematics is not to be demanded in all cases, but only in the case of things which have no matter. Hence its method is not that of natural science ; for presumably the whole of nature has matter.

If we are to appreciate the wisdom of these observations about the influence of temperament and of early training upon our standards of judgment concerning the value of explanation and proofs, we must recall what Aristotle had established regarding the nature of science in general, and of mathematical science in particular. Science unqualified was not intended to mean just true knowledge of any kind, but knowledge gained, not by mere calculation but by demonstration from first, self-evident, principles. In view of what we have already seen, it ought to be clear that the terms ' science,' ' demonstration,' ' first,' ' self-evident ' and ' principles ' cannot have anything in common with what they must evoke in the mind of Lord Russell. In fact, there can scarcely be a single term in Aristotelian philosophy whose meaning, as intended there, has anything in common (this word ' common ' not excepted) with the meaning that will arise once Russell has applied his canon of verification to it. The reader may feel that we exaggerate. But see what happens to ' existence ' and ' being ' in the following passage : " Since ' is ' does not belong to the primary language [for instead of 'A is yellow ' a logical language will say ' yellow (A) '], ' existence ' and ' being ' [as they occur in traditional metaphysics], if they are to mean anything, must be linguistic concepts not directly applicable to objects." [1] Lord Russell may at times appear to take up a word in the sense that we would intend. But even this appearance of a common significance soon vanishes. The word ' universal ' is a case in point. To him it reduces to ' similar,' and then ' similar ' becomes something that cannot, perhaps, be verified in a satisfactory way.[2] These remarks are not intended to be denigratory of Lord Russell, but only to remind ourselves how naive it would be to think that we could find some common ground with him outside what he calls logic. If such a common ground could ever be reached, it would probably prove to be unimportant.

There is a tempting advantage in Lord Russell's way of thinking. If his road be followed, the apparently simple questions raised by Aristotle and their extremely difficult answers need not be considered at all. At the very least, this should make for economy of thought. In the end, it might even come to pass that the only thought worthy of the name would be of the kind that may be fed to the computer. Lord Russell, I think, would not feel quite certain of this, but only definitely

1. *An Inquiry into Meaning and Truth*, New York, 1940, p.79. Some followers of PARMENIDES, suppressed the copula ' is ' from ' Man is white,' and confined themselves to ' White man.' *Physics*, I, 185 b 25.

2. *Op. cit.*, chap. XXV.

uncertain, which is again unavoidably obscure. In the present context, you see, certitude would have to be defined by way of interpretation, as in ' I certainly feel too warm,' which could be emotional and therefore not scientific. From this point of view, it can be shown to the satisfaction of any modern logical thinker that, in Euclid's construction of an equilateral triangle, the reason why the two circles intersect is really an emotional one : though muddle-headed in his assumptions he was determined to carry on nonetheless. So that the interpretation of Euclid's geometry really belongs to the province of behaviour, which is unscientific. This I believe to be a fair instance of the way Lord Russell reasons outside the domain where he is at home, and where nothing very definite can be expected since it lies outside the range of the computers. Would it be impertinent to suggest, as a final estimate, that the valid element in this sort of cogitation appears limited to what the machines would think concerning what they themselves are doing — if they could ?

Returning now to our passage from the *Metaphysics*, we can see that Aristotle would have held it impossible to teach anything whatsoever on any subject to a man who rejects every statement not endowed with the rigour (still doubtful, mind you, in some regard or other until the computer gets hold of it) of A(A). Must we grant that this stickler for precision is right ? Must we bow to his demand always and in every field of knowledge ? For that is what our problem amounts to. Aristotle would have said, not that we should change the subject, there being no subject left to change, but that we should not try to talk to him, seeing that his canon of exactness forbids him even to listen. It is bold of us to carry on anyhow, and perhaps optimistic to count on still having someone in the audience.

At any rate we must make ourselves aware of the extent to which all doors have been shut nowadays. The reader may be inclined to accuse us of exaggerating the claims of the computer-men to the point where they are represented as holding that the activity once called philosophy will never achieve the accuracy demanded by modern thought, nor ever extract itself from the quagmire of endless disputation, until the machines are called in to solve its problems — the possibility of feeding them to the Queen of Queens ' slaves being the very criterion of their relevance. Yet does Lord Russell assert anything less ?

Two hundred years ago, Leibniz foresaw the science which Peano has perfected, and endeavoured to create it. He was prevented from succeeding by respect for the authority of Aristotle, whom he could not believe guilty of definite, formal fallacies ; but the subject which he desired to create now exists, in spite of the patronising contempt with which his schemes have been treated by all superior persons. From this " Universal Characteristic," as he called it, he hoped for a solution of all problems, and an end to all disputes. " If controversies were to arise," he says, " there would

be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pens in their hands, to sit down to their desks, and to say to each other (with a friend as witness, if they liked), ' Let us calculate.' " This optimism has now appeared to be somewhat excessive ; there still are problems whose solution is doubtful, and disputes which calculation cannot decide. But over an enormous field of what was formerly controversial, Leibniz's dream has become sober fact. In the whole philosophy of mathematics, which used to be at least as full of doubt as any other part of philosophy, order and certainty have replaced the confusion and hesitation which formerly reigned. Philosophers, of course, have not yet discovered this fact, and continue to write on such subjects in the old way. But mathematicians, at least in Italy, have now the power of treating the principles of mathematics in an exact and masterly manner, by means of which the certainty of mathematics extends also to mathematical philosophy. Hence many of the topics which used to be placed among the great mysteries — for example, the natures of infinity, of continuity, of space, time and motion — are now no longer in any degree open to doubt or discussion. Those who wish to know the nature of these things need only read the works of such men as Peano or Georg Cantor ; they will there find exact and indubitable expositions of all these quondam mysteries.[1]

Have we no further choice? If we make one, we may as well realize that, to the mass of academic mankind, we are turning to what Russell calls " the vulgar prejudices of common sense." But the undertaking could prove worth while, if it did no more than teach us what those prejudices are and to what they lead.

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1. *Mysticism and Logic*, pp.79-80. Reprinted by permission of the publishers, Allen and Unwin.