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Complementarity Problems And Positive Definite Matrices

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Résumé de l'article

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Complementarity Problems And Positive Definite Matrices

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Abstract

The class of positive definite and positive semidefinite matrices is one of the most frequently encountered matrix classes both in theory and practice. In statistics, these matrices appear mostly with symmetry. However, in complementarity problems generally symmetry in not necessarily an accompanying feature. Linear complementarity problems defined by positive semidefinite matrices have some interesting properties such as the solution sets are convex and can be processed by Lemke's algorithm as well as Graves' principal pivoting algorithm. It is known that the principal pivotal transforms (PPTs) (defined in the context of linear complementarity problem) of positive semidefinite matrices are all positive semidefinite. In this article, we introduce the concept of generalized PPTs and show that the generalized PPTs of a positive semidefinite matrix are also positive semidefinite. One of the important characterizations of $\boldsymbol{P}$-matrices (that is, the matrices with all principle minors positive) is that the corresponding linear complementarity problems have unique solutions. In this article, we introduce a linear transformation and characterize positive definite matrices as the matrices with corresponding semidefinite linear complementarity problem having unique solutions. Furthermore, we present some simplification procedure in solving a particular type of semidefinite linear complementarity problems involving positive definite matrices.

Key words: Complementarity problems, Positive semidefinite matrices, principal pivotal transforms.

1. Introduction

A matrix $A \in \boldsymbol{R}^{n\times n}$ is said to be positive definite (positive semidefinite) if the quadratic form x^tAx is positive (nonnegative) for every nonzero $x \in \boldsymbol{R}^n$. The class of positive definite matrices is one of the most frequently encountered matrix classes both in theory and practice. The variance-covariance matrices encountered in Statistics are mostly *symmetric* positive definite and seldom positive semidefinite. However, unlike in many statistical applications, symmetry of the positive definite or semidefinite matrices is not an accompanying feature always. In linear complementarity problem (LCP) positive definiteness and positive semidefiniteness of matrices is defined without symmetry. Many characterizations of symmetric positive definite and semidefinite matrices are known. However, no direct results are known to be available on the characterization of positive definite and semidefinite matrices without going into symmetrization.

A number of matrix classes have been evolved while studying LCP. The class of $\boldsymbol{P}$-matrices (that is, real square matrices with all principal minors positive) is characterized in terms of LCP. This characterization says that a matrix A is a $\boldsymbol{P}$-matrix if, and only if, LCP (q, A) has a unique solution for every q [13]. In the

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context of LCPs with positive semidefinite matrices, it is known that solution sets of such problems are convex; and the problems can be processed by Lemke's algorithm as well as Graves' principal pivoting algorithm (see [2]). Furthermore, the principal pivotal transforms (PPTs) of a matrix defined in LCP are positive semidefinite provided the matrix is so. PPTs are defined only with respect to the nonsingular principal submatrices. In this article we introduce the concept of generalized PPTs using the Moore-Penrose inverse [11, 12] and show that even the generalized PPTs of positive semidefinite matrices are positive semidefinite. As a corollary to this result we deduce that the generalized Schur complement of a positive semidefinite matrix is also positive semidefinite.

An interesting result of this article is that we obtain a characterization of positive definite matrices that is similar to the one for $\boldsymbol{P}$-matrices. This characterization says that a matrix A is positive definite if, and only if, the solution to semidefinite linear complementarity problem $SDLCP(Q, AXA^t)$ has unique solution for every Q. The semidefinite linear complementarity problem (SDLCP) was introduced by Kojima, Shindoh and Hara as a unified model of various problems. LCP is a special case of SDLCP and many system and control theory problems as well as combinatorial optimization problems can be formulated as SDLCP (see [2, 4, 3, 8]).

Gowda and Song [6] extended a number of LCP concepts such as $\boldsymbol{R_0}$-property, $\boldsymbol{Q}$-property, $\boldsymbol{P}$-property and so on to SDLCP and studied their properties. In particular, they consider the Lyaponov transformation $L_A(X) = AX + XA^t$ and show that for this transformation, the $\boldsymbol{P}$-property and the $\boldsymbol{Q}$-property are equivalent to A being positive stable (that is, the real parts of eigen values of A are positive). Later Gowda and Parthasarathy [5] gave a characterization of positive stable matrices and deduced Stein's theorem using complementarity formulations. In this article we consider the linear transformation $L_A(X) = AXA^t$ and show that positive definiteness of A is equivalent to several properties of the SDLCP with respect to this linear transformation. In this context we also present some related results pertaining to the SDLCP with the proposed transformation. In addition, we present a closed form solution to the SDLCP with respect to the transformation AXA^t where A is any 2×2 positive definite matrix. For the general case on the order of A, we provide a simplification procedure.

The organization of this paper is as follows. In Section 2 we introduce the notation and the necessary preliminaries. In Section 3 we present the results related to LCP and some properties of positive semidefinite matrices. In Section 4 we present the results on positive definite matrices and the SDLCP with respect to the linear transformation AXA^t.

2. Notation and Preliminaries

Given a matrix $A \in \boldsymbol{R}^{n\times n}$ and $q \in \boldsymbol{R}^n$ the Linear Complementarity Problem (LCP) (q, A) is to find a vector $z \in \boldsymbol{R}^n$ such that

$$Az + q \geq 0, \quad z \geq 0 \text{ and } \; z^t(Az + q) = 0. \quad (1)$$

The reader may refer [2, 9] for a detailed account of LCP and its applications. We shall use the notation $S(q, A)$ to denote the set of solutions to (q, A). A number of matrix classes have been evolved while studying LCP. We shall recall the definitions of some classes relevant to this article. The class of $\boldsymbol{Q}$-matrices consists of all those matrices A for which $S(q, A)$ is nonempty for every q. Unless stated otherwise all the matrices in this paper are to be treated as real square matrices and all vectors as real vectors. A matrix A is said to be positive definite (positive semidefinite) if the quadratic form x^tAx is positive (nonnegative) for every nonzero vector x of appropriate order. The class of $\boldsymbol{P}$-matrices includes all matrices whose principal minors are all positive. It is a well known fact that the positive definite matrices are in $\boldsymbol{P}$ and the principal minors of any positive semidefinite matrix are all nonnegative.

The principal pivotal transform (PPT) of any matrix A with respect to a nonsingular principal submatrix $A_{\alpha\alpha}$ of A is defined as the matrix M where

$$\begin{aligned} M_{\alpha\alpha} &= (A_{\alpha\alpha})^{-1}, \\ M_{\alpha\bar{\alpha}} &= -M_{\alpha\alpha}A_{\alpha\bar{\alpha}}, \\ M_{\bar{\alpha}\alpha} &= A_{\bar{\alpha}\alpha}M_{\alpha\alpha}, \\ M_{\bar{\alpha}\bar{\alpha}} &= A_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha}A_{\alpha\bar{\alpha}}. \end{aligned}$$

Here α stands for an index set and $\bar{\alpha}$ for its complement. It may be verified that the PPT of M with respect to α is A itself. For the notation see [2]. PPTs play an important role in LCP and have very interesting properties. Many properties are closed under PPTs. For example, if A is a $\boldsymbol{P}$-matrix, then all its PPTs are also $\boldsymbol{P}$-matrices. This result holds good for positive definite, positive semidefinite and $\boldsymbol{Q}$-matrices as well. For more elaborate discussion and results on PPTs refer to [2, 9]. It must be noted that PPTs are defined only with respect to the nonsingular principal submatrices of the given matrix. In this article we define PPTs even with respect to any singular principal submatrix using its Moore-Penrose generalized inverse which is defined as follows.

Definition 1. A matrix B is called the Moore-Penrose inverse of a matrix A provided the following four conditions are satisfied:

(i) $ABA = A$
(ii) $BAB = B$
(iii) $(AB)^t = AB$
(iv) $(BA)^t = BA.$

The Moore-Penrose inverse of A is denoted by A^+. Moore-Penrose inverse exists and is unique. Furthermore, the Moore-Penrose inverse of Moore-Penrose inverse is the original matrix, that is, $(A^+)^+ = A$ (see [11]). In Section 3 we will show that Moore-Penrose inverse of any positive semidefinite matrix is also positive semidefinite.

We now present the semidefinite linear complementarity problem (SDLCP). Let $\mathcal{S}^n$ denote the class of real symmetric matrices of order n. For any matrix X, not necessarily symmetric, we use the notation $X \succ 0$ $(X \succeq 0)$ to indicate that X is positive definite (positive semidefinite). Given any linear transformation L from $\mathcal{S}^n$ to $\mathcal{S}^n$ and a $Q \in \mathcal{S}^n$, the SDLCP is to find an $X \in \mathcal{S}^n$ such that

$$Y = L(X) + Q \succeq 0, \quad X \succeq 0, \text{ and } XY = 0. \qquad (2)$$

We shall denote this problem by $SDLCP(Q, L)$. In the above equation, X is called a solution of $SDLCP(Q, L)$. In connection with the SDLCP we need the following preliminaries.

Let $A,\ B \in \boldsymbol{R}^{n\times n}$ be any arbitrary matrices. Then

(1) trace of A is defined as the sum of its diagonal entries and is denoted by $tr(A)$;
(2) $tr(AB) = tr(BA)$;
(3) A is said to be orthogonal if $AA^t = A^tA = I$, the identity matrix;
(4) if A is symmetric matrix, then there exists an orthogonal matrix P such that PAP^t is a diagonal matrix with real entries;
(5) if A and B commute (that is, AB=BA), then there exists an orthogonal matrix P such that PAP^t and PBP^t are real diagonal matrices;
(6) if A is positive definite (positive semidefinite), then $tr(A) > 0,\ (tr(A) \geq 0)$;
(7) if A is symmetric positive semidefinite, then $tr(A) = 0$ if, and only if, $A = 0$;
(8) if A and B are symmetric positive semidefinite, then $tr(AB) = 0$ if, and only if, $AB = BA = 0$.

3. LCP And Positive Semidefinite Matrices

In this section we introduce the concept of generalized PPTs and show that if a matrix is positive semidefinite, then so are its generalized PPTs. In addition, we present some characterizations of positive semidefinite matrices, not necessarily symmetric, and study other properties of these matrices.

In the previous section we defined the PPTs of a given matrix. In order to define the PPT of a matrix with respect to a principal submatrix, the principal submatrix must be nonsingular. Given a LCP, we can transform this problem into another LCP in which the data matrix is the PPT of the original matrix. The transformed LCP is equivalent to the original one in the sense that there is a one to one correspondence between the solutions of the new and original LCPs. However, in the case of singular principal submatrices, the concept of PPTs does not exist. It may be worth defining the PPTs with reference to the singular principal submatrices using their generalized inverses. Here we use the Moore-Penrose inverse to define the PPTs as this leads us to some kind of equivalence of the LCPs under some special conditions (see Theorem 3). Let us first define the generalized PPT of a matrix with respect to any of its singular principal submatrices.

Definition 2. Let $A \in \boldsymbol{R}^{n\times n}$ and let $A_{\alpha\alpha}$ be a principal submatrix of A, not necessarily nonsingular. The generalized PPT (GPPT) of A with respect to $A_{\alpha\alpha}$ is defined by

$$\begin{aligned} M_{\alpha\alpha} &= (A_{\alpha\alpha})^+, \\ M_{\alpha\bar{\alpha}} &= -M_{\alpha\alpha}A_{\alpha\bar{\alpha}}, \\ M_{\bar{\alpha}\alpha} &= A_{\bar{\alpha}\alpha}M_{\alpha\alpha}, \\ M_{\bar{\alpha}\bar{\alpha}} &= A_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha}A_{\alpha\bar{\alpha}}. \end{aligned}$$

Since $(A_{\alpha\alpha})^+$ is unique, the GPPT is uniquely defined, and when $A_{\alpha\alpha}$ is nonsingular GPPT coincides with the usual PPT.

Earlier it was mentioned that PPT of PPT of a matrix with respect to the same index set is the matrix itself. But this is not the case with the GPPTs. The following theorem presents the conditions under which the GPPT of GPPT of matrix with respect to the same index set is the original matrix.

Theorem 3. Let $A \in \boldsymbol{R}^{n\times n}$ and let M be its GPPT with respect to some index set α. Let B be the GPPT of M with respect to α. The necessary and sufficient conditions for B to be equal to A are given by

(i) column span of $A_{\alpha\bar{\alpha}}$ is contained in the column span of $A_{\alpha\alpha}$, and

(ii) row span of $A_{\bar{\alpha}\alpha}$ is contained in the row span of $A_{\alpha\alpha}$.

Proof. We have

$$M = \begin{bmatrix} A^+_{\alpha\alpha} & -A^+_{\alpha\alpha}A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha}A^+_{\alpha\alpha} & A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}A^+_{\alpha\alpha}A_{\alpha\bar{\alpha}} \end{bmatrix}.$$

Using the fact $(A^+_{\alpha\alpha})^+ = A_{\alpha\alpha}$ it may be verified that

$$B = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\alpha}A^+_{\alpha\alpha}A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha}A^+_{\alpha\alpha}A_{\alpha\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

It is easy to check that for $A_{\alpha\alpha}A^+_{\alpha\alpha}A_{\alpha\bar{\alpha}}$ to be equal to $A_{\alpha\bar{\alpha}}$ a necessary and sufficient condition is that $A_{\alpha\bar{\alpha}}$ is contained in the column span of $A_{\alpha\alpha}$. Similarly, the necessary and sufficient condition for $A_{\bar{\alpha}\alpha}A^+_{\alpha\alpha}A_{\alpha\alpha}$ to be equal to $A_{\bar{\alpha}\alpha}$ is that the row span of $A_{\bar{\alpha}\alpha}$ is contained in the row span of $A_{\alpha\alpha}$. □

Corollary 4. Let $A \in \boldsymbol{R}^{n\times n}$ and let M be its GPPT with respect to some index set α. Let B be the GPPT of M and let C be the GPPT of B with respect to α. Then $C = M$.

Proof. Consider B given in the proof of the theorem. It is clear that $A_{\alpha\alpha}A^+_{\alpha\alpha}A_{\alpha\bar{\alpha}}$ is contained in the column span of $A_{\alpha\alpha}$ and $A_{\bar{\alpha}\alpha}A^+_{\alpha\alpha}A_{\alpha\alpha}$ is contained in the row span of $A_{\alpha\alpha}$. It follows that $C = M$. □

The conditions (i) and (ii) of Theorem 3 automatically hold good when A is a symmetric positive semidefinite matrix. This amounts to Albert's theorem [1] (also see Theorem 8.8.3, pp.321 of [11]).

We shall now look at some results on positive semidefinite matrices. The following result is needed in the sequel.

Theorem 5. If A is positive semidefinite, then it is range symmetric, that is, its column span and the row span are the same.

Proof. Suffices to show that for all $x \in \boldsymbol{R}^n$, $Ax = 0$ implies $A^t x = 0$. Let $x \in \boldsymbol{R}^n$ be such that $Ax = 0$. Then

$$x^t(A + A^t)x = 0 \tag{3}$$

Since A is positive semidefinite, $A + A^t$ is symmetric positive semidefinite and hence $A + A^t = C^tC$ for some matrix C. From (3) it follows that $x^tC^tCx = 0$ which implies $Cx = 0$. This in turn implies $C^tCx = 0$ or $(A + A^t)x = 0$. Thus, we have $A^tx = -Ax = 0$. □

Our next result is a characterization of positive semidefinite matrices.

Theorem 6. Let $A \in \boldsymbol{R}^{n\times n}$ with rank r. Then A is positive semidefinite if, and only if, it can be expressed as $A = BTB^t$ where B is a $n \times r$ semiorthogonal matrix (that is, $B^tB = I$) and T is a $r \times r$ nonsingular positive semidefinite matrix.

Proof. 'If part' is obvious. We shall prove the only if part. Let r be the rank of A and let B be an $n \times r$ matrix whose columns form an orthonormal basis (i.e., $B^tB = I$) for the column span of A. This means there exists an $r \times n$ matrix C such that $A = BC$. From Theorem 5, there also exists an $r \times n$ matrix D such that $A^t = BD$. Hence, we have $BC = D^tB^t$. Since B is of full column rank, we can write $C = GD^tB^t$ for some $r \times n$ matrix G. Let $T = GD^t$. Then $A = BC = BTB^t$. Note that $r = \text{rank}(A) \leq \text{rank}(T)$. Since T is an $r \times r$ matrix, it must be nonsingular. Thus, $A = BTB^t$ where $B^tB = I$ and T is nonsingular. To complete the proof, we need to show that T is positive semidefinite. But this immediately follows from the fact that $T = B^tAB$ and the hypothesis that A is positive semidefinite. □

Earlier we have mentioned that PPTs of positive semidefinite matrices are also positive semidefinite. We shall prove that this is true even with the GPPTs. First, we shall prove that the Moore-Penrose inverse of a positive semidefinite matrix is positive semidefinite.

Theorem 7. Let $A \in \boldsymbol{R}^{n\times n}$. The following hold:

(i) if A is positive semidefinite with rank r, then there exist an $n \times r$ matrix B and an $r \times r$ nonsingular matrix T such that $B^tB = I$, $A = BTB^t$ and $A^+ = BT^{-1}B^t$,

(ii) A is positive semidefinite if, and only if, A^+ is so,

(iii) if A is positive semidefinite, then A and A^+ commute, that is $AA^+ = A^+A$.

Proof.

(i): The existence of B and T satisfying the conditions was already established in Theorem 6. It can be checked that $BT^{-1}B^t$ is indeed equal to A^+. Thus, $A^+ = BT^{-1}B^t$.

(ii): Follows from the fact that T is positive semidefinite if, and only if, T^{-1} is so.

(iii): This is a direct consequence of (i) and (ii). □

We will now show that GPPT of any positive semidefinite matrix is also positive semidefinite.

Theorem 8. Let $A \in \boldsymbol{R}^{n\times n}$ be a positive semidefinite matrix and let α be an arbitrary index set. Then the GPPT M of A with respect to $A_{\alpha\alpha}$ is positive semidefinite.

Proof. Let z be an arbitrary column vector in $\boldsymbol{R}^n$ and let $w = Az$. Premultiplying both sides of $w = Az$ with

$\begin{bmatrix} -A^+_{\alpha\alpha} & 0 \\ -A_{\bar\alpha\alpha}A^+_{\alpha\alpha} & I_{\bar\alpha\bar\alpha} \end{bmatrix}$ and rearranging the terms it can be shown that

$$\begin{bmatrix} 0 & A^+_{\alpha\alpha}A_{\alpha\alpha} \\ I & -A_{\bar\alpha\alpha}(I - A^+_{\alpha\alpha}A_{\alpha\alpha}) \end{bmatrix} \begin{bmatrix} w_{\bar\alpha} \\ z_\alpha \end{bmatrix} = M \begin{bmatrix} w_\alpha \\ z_{\bar\alpha} \end{bmatrix} \quad (4)$$

Now premultiplying both sides of (4) with $\begin{bmatrix} w_\alpha \\ z_{\bar\alpha} \end{bmatrix}^t$ and simplifying we get

$$(w^t_\alpha,\ z^t_{\bar\alpha})M \begin{bmatrix} w_\alpha \\ z_{\bar\alpha} \end{bmatrix} = z^t_{\bar\alpha}w_{\bar\alpha} + w^t_\alpha A^+_{\alpha\alpha}A_{\alpha\alpha}z_\alpha - z^t_{\bar\alpha}A_{\bar\alpha\alpha}(I - A^+_{\alpha\alpha}A_{\alpha\alpha})z_\alpha. \quad (5)$$

Now,

$$\begin{aligned} w^t_\alpha A^+_{\alpha\alpha}A_{\alpha\alpha}z_\alpha &= (z^t_\alpha A^t_{\alpha\alpha} + z^t_{\bar\alpha}A^t_{\alpha\bar\alpha})A^+_{\alpha\alpha}A_{\alpha\alpha}z_\alpha \\ &= z^t_\alpha A^t_{\alpha\alpha}A^+_{\alpha\alpha}A_{\alpha\alpha}z_\alpha \\ &\quad + z^t_{\bar\alpha}A^t_{\alpha\bar\alpha}A^+_{\alpha\alpha}A_{\alpha\alpha}z_\alpha \\ &= z^t_\alpha A^t_{\alpha\alpha}z_\alpha + z^t_{\bar\alpha}A^t_{\alpha\bar\alpha}A^+_{\alpha\alpha}A_{\alpha\alpha}z_\alpha \\ &= z^t_\alpha A_{\alpha\alpha}z_\alpha + z^t_{\bar\alpha}A^t_{\alpha\bar\alpha}A^+_{\alpha\alpha}A_{\alpha\alpha}z_\alpha \end{aligned}$$

Notice that the last equation holds because $A^t_{\alpha\alpha}A^+_{\alpha\alpha}A_{\alpha\alpha} = A^t_{\alpha\alpha}$ as row span and column span of $A_{\alpha\alpha}$ are the same as $A_{\alpha\alpha}$ is positive semidefinite.

Now right hand side of (5) can be written as

$$\begin{aligned} & z^t_{\bar\alpha}w_{\bar\alpha} + z^t_\alpha w_\alpha - z^t_{\bar\alpha}A^t_{\alpha\bar\alpha}z_\alpha + z^t_{\bar\alpha}A^t_{\alpha\bar\alpha}A^+_{\alpha\alpha}A_{\alpha\alpha}z_\alpha \\ & -z^t_{\bar\alpha}A_{\bar\alpha\alpha}(I - A^+_{\alpha\alpha}A_{\alpha\alpha})z_\alpha \\ & = z^tAz - z^t_{\bar\alpha}(A^t_{\alpha\bar\alpha} + A_{\bar\alpha\alpha})(I - A^+_{\alpha\alpha}A_{\alpha\alpha})z_\alpha \\ & = z^tAz \geq 0 \end{aligned} \quad (6)$$

(Notice that since A is positive semidefinite, $A^t_{\alpha\bar\alpha} + A_{\bar\alpha\alpha} = D(A_{\alpha\alpha} + A^t_{\alpha\alpha}) = RA_{\alpha\alpha}$ for some D and R as rank of $A_{\alpha\alpha}$ is equal to that of $A^t_{\alpha\alpha}$).

Thus

$$\begin{bmatrix} w_\alpha \\ z_{\bar\alpha} \end{bmatrix}^t M \begin{bmatrix} w_\alpha \\ z_{\bar\alpha} \end{bmatrix} = z^tAz\ \forall z \in \boldsymbol{R}^n$$
$$\text{and } w_\alpha = A_{\alpha\alpha}z_\alpha + A_{\alpha\bar\alpha}z_{\bar\alpha} \geq 0. \quad (7)$$

Now, let u be an arbitrary vector in $\boldsymbol{R}^n$. Since $[A_{\alpha\alpha}\ :\ (I - A^+_{\alpha\alpha}A_{\alpha\alpha})]$ is of full row rank (this is because $A^t_{\alpha\alpha}(I - A^+_{\alpha\alpha}A_{\alpha\alpha}) = 0$ and rank $(A_{\alpha\alpha})$ + rank$(I - A^+_{\alpha\alpha}A_{\alpha\alpha}$)=$|\alpha|$), there exist a $v, g \in \boldsymbol{R}^{|\alpha|}$ such that

$$u_\alpha - A_{\alpha\bar\alpha}u_{\bar\alpha} = A_{\alpha\alpha}v + (I - A^+_{\alpha\alpha}A_{\alpha\alpha})g. \quad (8)$$

Taking $z_\alpha = v,\ z_{\bar\alpha} = u_{\bar\alpha}$ and rewriting (8), we get

$$\begin{aligned} u_\alpha &= A_{\alpha\alpha}z_\alpha + A_{\alpha\bar\alpha}z_{\bar\alpha} + (I - A^+_{\alpha\alpha}A_{\alpha\alpha})g \\ &= w_\alpha + (I - A^+_{\alpha\alpha}A_{\alpha\alpha})g. \end{aligned} \quad (9)$$

Now,

$$\begin{aligned} u^tMu &= \begin{bmatrix} u_\alpha \\ z_{\bar\alpha} \end{bmatrix}^t M \begin{bmatrix} u_\alpha \\ z_{\bar\alpha} \end{bmatrix} \\ &= \begin{bmatrix} w_\alpha + (I - A^+_{\alpha\alpha}A_{\alpha\alpha})g \\ z_{\bar\alpha} \end{bmatrix}^t \times \\ &\quad M \begin{bmatrix} w_\alpha + (I - A^+_{\alpha\alpha}A_{\alpha\alpha})g \\ z_{\bar\alpha} \end{bmatrix} \\ &= \begin{bmatrix} w_\alpha \\ z_{\bar\alpha} \end{bmatrix}^t M \begin{bmatrix} w_\alpha \\ z_{\bar\alpha} \end{bmatrix} \geq 0 \text{ from (7).} \end{aligned}$$

As u is any arbitrary vector, it follows that M is positive semidefinite. □

A number of papers have been written on Schur complements and their applications, particularly in the context of positive semidefinite matrices. For statistical applications and bibliography on Schur complements and its generalizations see [10, 14].

Corollary 9. Let $A \in \boldsymbol{R}^{n\times n}$ be positive semidefinite and let α be an arbitrary index set. The generalized Schur complement of A with respect to $A_{\alpha\alpha}$ is positive semidefinite.

4. SDLCP and Positive Definite Matrices

In this section we study some properties of SDLCP with respect to a special linear map $L(X) = AXA^t$. Using this we present a characterization of positive definite matrices. We show that solving SDLCP with this special linear map in which A is positive definite is reduced to solving a quadratic equation in matrices with a constraint. Under some special cases the quadratic equation reduces to linear equation.

Let $L : \mathcal{S}^n \to \mathcal{S}^n$ be a linear map and let $Q \in \mathcal{S}^n$. Consider the problem $SDLCP(Q, L)$. Gowda and Song [6] introduced the following concepts in SDLCP which are along the lines of similar concepts in LCP.

Definition 10. The map L is said to be $\boldsymbol{R_0}$ if $SDLCP(0, L)$ has a unique solution; L is said to be a $\boldsymbol{Q}$-map if the $SDLCP(Q, L)$ has a solution for every $Q \in \mathcal{S}^n$; L is said to have GUS property if $SDLCP(Q, L)$ has a unique solution for every $Q \in \mathcal{S}^n$.

Definition 11. L is said to have P-property if the following implication holds for every $X \in \mathcal{S}^n$:

$$X \text{ and } L(X) \text{ commute, } XL(X) \preceq 0 \Rightarrow X = 0.$$

L is said to have $\boldsymbol{P_1}$-property if the following implication holds for every $X \in \mathcal{S}^n$:

$$XL(X) \preceq 0 \Rightarrow X = 0.$$

Notice that $\boldsymbol{P_1}$-property implies $\boldsymbol{P}$-property. In the sequel we need the following result which can be deduced as a special case from Karamardian's result [7].

Theorem 12. If $L \in \boldsymbol{R_0}$ and $SDLCP(Q, L)$ has a unique solution for some positive definite matrix $Q \in \mathcal{S}^n$, then L is a $\boldsymbol{Q}$-map.

Let $A \in \boldsymbol{R}^{n\times n}$. We shall now study the various properties of the linear transformation $L_A : \mathcal{S}^n \to \mathcal{S}^n$ defined by

$$L_A(X) = AXA^t, \quad X \in \mathcal{S}^n. \tag{10}$$

Throughout this section we shall use the notation L_A for the linear transformation defined in (10) induced by the matrix A. We shall call this transformation as **multiplicative transformation**.

First we shall present two propositions which show how SDLCPs with L_A can be simplified.

Proposition 13. Let $A \in \boldsymbol{R}^{n\times n}$ and consider the $SDLCP(Q, L_A),\ Q \in \mathcal{S}^n$. This problem can be reduced to another equivalent $SDLCP(D, L_M)$ where D is a diagonal matrix and $M = PAP^t$ for some orthogonal matrix P.

Proof. Since Q is a real symmetric matrix, its eigen values are real and there exists an orthogonal matrix P such that $D = PQP^t$ is a diagonal matrix. By premultiplying and post multiplying the equation $Y = AXA^t + Q$ with P and P^t respectively, we get

$$\begin{aligned} PYP^t &= PAP^tPXP^tPA^tP^t + PQP^t \text{ or} \\ W &= MZM^t + D, \end{aligned} \tag{11}$$

where $M = PAP^t$, $W = PYP^t$ and $Z = PXP^t$. Clearly, $WZ = ZW = 0$ if, and only if, $XY = YX = 0$. Thus, the $SDLCP(Q, AXA^t)$ is equivalent to $SDLCP(D, MXM^t)$ in which D is diagonal and $M = PAP^t$ for some orthogonal matrix P. □

Proposition 14. Let $A \in \mathcal{S}^n$ and consider the $SDLCP(Q, L_A),\ Q \in \mathcal{S}^n$. This problem can be reduced to an equivalent $SDLCP(H, L_M)$ in which M is a real diagonal matrix.

Proof. Follows from the fact that, since A is symmetric, there exists an orthogonal matrix P such that PAP^t is diagonal. □

The following proposition states that if L_A has $\boldsymbol{P}$-property or $\boldsymbol{Q}$-property, then A must necessarily be nonsingular.

Proposition 15. Let $A \in \boldsymbol{R}^{n\times n}$ and consider L_A. If L_A has either $\boldsymbol{P}$-property or $\boldsymbol{Q}$-property, then A is nonsingular.

Proof. Assume L_A has $\boldsymbol{P}$-property. Suppose A is singular. Then there is an nonzero $x \in \boldsymbol{R}^n$ such that $Ax = 0$. Let $X = xx^t$. Then $X \neq 0$ and $AX = 0$. Hence $XL_A(X) = L_A(X)X = 0$. This violates the $\boldsymbol{P}$-property. It follows that A is nonsingular.

Next assume that L_A has $\boldsymbol{Q}$-property. Let X be a solution of $SDLCP(-I,\ L_A)$. Then it follows that $AXA^t = Y + I \succ 0$ for some $Y \succeq 0$. This in turn implies $AXA^t \succ 0$ and hence A is nonsingular. □

Gowda and Song [6] presented a number of equivalent conditions with respect to the Lyapunov transformation given by $L(X) = AX + XA^t$. In the following theorem we present a number of equivalent conditions with respect to L_A. Earlier it was mentioned that LCP provides a complete characterization of $\boldsymbol{P}$-matrices. It says that a matrix A is a $\boldsymbol{P}$-matrix if, and only if, LCP (q, A) has a unique solution for every q. The theorem below presents a similar characterization of positive definite matrices. It says that a matrix A is positive definite if, and only if, $SDLCP(Q,\ L_A)$ has a unique solution for every Q. We need the following lemma in the sequel.

Lemma 16. Suppose A is a symmetric positive semidefinite matrix. Then for any positive semidefinite matrix B the $tr(BA)$ is nonnegative.

Proof. Since A is symmetric positive semidefinite, there exists a matrix U such that $A = UU^t$. We have $tr(BA) = tr(BUU^t) = tr(U^tBU)$. Since B is positive semidefinite, so is U^tBU and hence $tr(U^tBU)$ is nonnegative. Thus, $tr(BA)$ is nonnegative. □

Theorem 17. Let $A \in \boldsymbol{R}^{n\times n}$. The following conditions are equivalent:

(*i*) A is either positive definite or negative definite.

(*ii*) For every $Q \in \mathcal{S}^n,\ SDLCP(Q,\ L_A)$ has at most one solution.

(*iii*) L_A has GUS property.

(*iv*) L_A has $\boldsymbol{P}$-property.

(*v*) L_A is $\boldsymbol{R_0}$.

(*vi*) $XAX = 0$ implies $X = 0$.

Proof.

To show that (i) implies (ii) we may assume that A is positive definite. We can make this assumption without loss of generality as $L_A(X) = L_{-A}(X)\ \forall X$. Let $Q \in \mathcal{S}^n$. Suppose X and Z are two solutions of $SDLCP(Q,\ L_A)$. We will show that $X = Z$. We have

$$Y = AXA^t + Q,\ X \succeq 0,\ Y \succeq 0,\ XY = YX = 0, \tag{12}$$

and

$$W = AZA^t + Q,\ Z \succeq 0,\ W \succeq 0,\ WZ = ZW = 0. \tag{13}$$

Subtracting (12) from (13), we get

$$W - Y = A(Z - X)A^t \tag{14}$$

Postmultiplying (14) with $(Z - X)$ and $(Z + X)$ separately and using (12) and (13) we get

$$-WX - YZ = A(Z - X)A^t(Z - X) \tag{15}$$

and

$$WX - YZ = A(Z - X)A^t(Z + X). \tag{16}$$

Adding (15) and (16) we get

$$-YZ = A(Z - X)A^tZ. \tag{17}$$

Premultiplying (17) with $(Z - X)$ and using $XY = 0$ we get

$$-ZYZ = (Z - X)A(Z - X)A^tZ. \tag{18}$$

Postmultiplying (18) with A, we get

$$-ZYZA = (Z - X)A(Z - X)A^tZA. \tag{19}$$

Observe that ZYZ and A^tZA are symmetric positive semidefinite matrices, and A and $(Z-X)A(Z-X)$ are positive and positive semidefinite matrices respectively. Applying Lemma 16, we find that $tr(-ZYZA) \leq 0$ and $tr((Z - X)A(Z - X)A^tZA) \geq 0$. Hence $tr(ZYZA) = 0$. This in turn implies $tr(ZYZ(A + A^t)) = 0$ and hence $ZYZ(A + A^t) = 0$. This implies $ZYZ = 0$ as A, and hence $(A + A^t)$, is positive definite. As Y is symmetric positive semidefinite, it follows that $YZ = 0$. Similarly, we can show that $WX = 0$. From (15) we conclude that $A(Z-X)A^t(Z-X) = 0$. As A is positive definite it follows that $Z - X = 0$ and hence $Z = X$. Thus, $SDLCP(Q,\ L_A)$ has at most one solution.

Since 0 is a solution of both $SDLCP(0,\ L_A)$ and $SDLCP(I,\ L_A)$, from Theorem 12 it follows that (ii) implies (iii).

The implication of (iv) from (iii) holds for any general linear map (see [6]). Implication of (v) from (iv) is also obvious.

Next, assume, if possible, that (vi) does not hold. Then there must exist a nonzero vector $x \in \boldsymbol{R}^n$ such that $x^tAx = 0$. Let $X = xx^t$. Clearly $X \succeq 0,\ AXA^t \succeq 0$ and $XAXA^t = 0$. Note that $X \neq 0$ as $x \neq 0$. Thus X is a nontrivial solution of $SDLCP(0, L_A)$. This contradicts the hypothesis that L_A is $\boldsymbol{R_0}$. It follows that (v) implies (vi).

To complete the proof of the theorem we will show that (vi) implies (i). Assume that (vi) holds and that A is not negative definite. We will show that A is positive definite. Assume that A is not positive definite. As A is not negative definite, there must exist vectors x and y in $\boldsymbol{R}^n$ such that $x^tAx < 0$ and $y^tAy > 0$. This in turn implies that a convex combination z of x and y satisfies $z^tAz = 0$. Clearly, $z \neq 0$. Letting $X = zz^t$, we obtain $XAX = 0$ and $X \neq 0$. It follows that (vi) implies (i). $\square$

When A is a positive definite matrix, the rank of any solution X of $SDLCP(Q,\ L_A)$ must be equal to the number of negative eigen values of Q. This is elaborated in our next result.

Theorem 18. Let $A \in \boldsymbol{R}^{n\times n}$ be a positive definite matrix and let X be a solution of $SDLCP(Q,\ L_A)$. Then rank of X is equal to the number of negative eigen values of Q. Furthermore, the rank of Y is equal to the number of positive eigen values of Q where $Y = AXA^t + Q$.

Proof. In view of Proposition 13, we may assume, without loss of generality, that $Q = \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & -\Delta_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where Δ_1 and Δ_2 are positive diagonal matrices. Assume that Δ_2 is an $s \times s$ matrix. Note that s is the number of negative eigen values of Q. Let $X \in \mathcal{S}^n$ be a solution of $SDLCP(Q, L_A)$. Let $Y = AXA^t + Q$. Assume that rank of X is r. We can find an $n \times r$ matrix U such that $X = UU^t$. Since $XYX = 0$ and since U is an $n \times r$ matrix with rank r, it follows that $U^tAUU^tA^tU + U^tQU = 0$. Since A is positive definite and U is of rank r, it follows that $U^tAUU^tA^tU$ is positive definite and hence $-U^tQU$ is positive definite. This implies $U_2^t\Delta_2U_2 - U_1^t\Delta_1U_1$ is positive definite and hence $U_2^t\Delta_2U_2$ is positive definite. Since U_2 is an $s \times r$ matrix, it follows that $s \geq r$. On the other hand, to see that $s \leq r$, let A_2 be the submatrix of A such that $A_2UU^tA_2^t$ is the principal submatrix of $Y = AUU^tA^t + Q$ corresponding to $-\Delta_2$ of Q. Since Y is positive semidefinite, so is its principal submatrix $A_2UU^tA_2^t - \Delta_2$. As Δ_2 is positive diagonal matrix, it follows that $A_2UU^tA_2^t$ is positive definite. This in turn implies rank of $A_2UU^tA_2^t(= s)$ is less than or equal to rank of $UU^t(= r)$. Therefore, rank of X is equal to the number of negative eigen values of Q.

Let $T = -A^{-1}Q(A^{-1})^t$. Observe that X is a solution of $SDLCP(Q, L_A)$ if, and only if, Y is a solution of $SDLCP(T, L_{A^{-1}})$. Note that the number of negative eigen values of T is equal to the number of positive eigen values of Q. From what we have shown above, it follows that rank of Y is equal to the number of positive eigen values of Q. □

In the rest of this paper we shall present some simplification procedure for solving SDLCP with L_A where A is positive definite. When the order of the matrix A is 2×2, we have a closed form solution, and in the general case we reduce the problem to that of solving some quadratic equation. We need the following algebraic facts in the sequel.

Fact 1. If A is positive definite and U is a full column rank matrix such that AU is defined, then U^tAU is positive definite.

Fact 2. If B is positive semidefinite and $A - B$ is positive definite, then A is positive definite.

Fact 3. Let B and C be matrices of order $m \times n$ such that $B^tB = C^tC$. Then $B = PC$ for some orthogonal matrix P.

Consider $SDLCP(Q,\ L_A)$. In view of Proposition 13, we can assume, without loss of generality, that $Q = \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & -\Delta_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where Δ_1 and Δ_2 are positive diagonal matrices. Let k be the order of Δ_1 and r be the order of Δ_2. We have already noted that rank of X is r. Let $X = UU^t$ be rank factorization of X for some $n \times r$ matrix U of rank r. Partition U as $(U_1^t,\ U_2^t,\ U_3^t)^t$ where U_2 is of order $r \times r$. Since A is positive definite and U is of full column rank, U^tAU is positive definite (Fact 1) and $U^tAUU^tA^tU$ is symmetric positive definite. Now,

$$\begin{aligned} U^tAUU^tA^tU &= U^tAXA^tU \\ &= -U^tQU \\ &= U_2^t\Delta_2U_2 - U_1^t\Delta_1U_1 \text{ is positive definite.} \end{aligned}$$

From Fact 2 it follows that U_2 is nonsingular and hence $U_1 = TU_2$ for some T of order $k \times r$. Thus $U^tAUU^tA^tU = U_2^t(\Delta_2 - T^t\Delta_1T)U_2$ where $\Delta_2 - T^t\Delta_1T$ is symmetric positive definite.

So, from Fact 3, it follows that $U^tA^tU = P(\Delta_2 - T^t\Delta_1T)^{\frac{1}{2}}U_2$ for some orthogonal matrix P.

Since $XAXA^t + XQ = 0$, we have

$$U^tAUU^tA^tU = -U^tQ \text{ or}$$

$$AUU^tA^tU = -QU = \begin{bmatrix} -\Delta_1U_1 \\ \Delta_2U_2 \\ 0 \end{bmatrix}.$$

Hence

$$AU = \begin{bmatrix} -\Delta_1U_1 \\ \Delta_2U_2 \\ 0 \end{bmatrix}(U^tA^tU)^{-1}. \tag{20}$$

Partitioning $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ where A_{11} and A_{22} are of orders $k \times k$ and $r \times r$ respectively, we can deduce the following from (20):

$$U_3 = -A_{33}^{-1}(A_{32} + A_{31}T)U_2, \tag{21}$$

$$U_2 = -(M_{21}T + M_{22})^{-1}\Delta_2(\Delta_2 - T^t\Delta_1T)^{-\frac{1}{2}}P^t, \tag{22}$$

and

$$M_{11}T + M_{13} + T\Delta_2^{-1}M_{21}T + T\Delta_2^{-1}M_{22} = 0, \tag{23}$$

where M is the PPT of A with respect to A_{33}.

Once (23) is solved for T subject to the condition that $\Delta_2 - T^t\Delta_1T$ is positive definite, then we can directly compute X without actually computing P. We shall now examine some special cases in which Q is assumed to be nonsingular.

Case 1. $Q = \begin{bmatrix} \Delta_1 & 0 \\ 0 & -\Delta_2 \end{bmatrix}$ and $A_{12} = 0$.

In this case we can deduce that $U_1^tA_{11}U_1 = -U_1^t\Delta_1U_1(U^tA^tU)^{-1}$ and using trace arguments, we can deduce that $U_1 = 0$. Further, $U_2U_2^t = A_{22}^{-1}\Delta_2A_{22}^t$. Therefore, $X = \begin{bmatrix} 0 & 0 \\ 0 & A_{22}^{-1}\Delta_2A_{22}^t \end{bmatrix}$ is a solution in this case.

Case 2. $Q = \begin{bmatrix} \Delta_1 & 0 \\ 0 & -\Delta_2 \end{bmatrix}$ and $A_{21} = 0$.

In this case, we can show that

$$U_2 = A_{22}^{-1}\Delta_2(\Delta_2 - T^t\Delta_1T)^{-\frac{1}{2}}P^t,$$

where P is an orthogonal matrix and T is the unique solution of

$$\Delta_1^{-1}A_{11}T + T\Delta_2^{-1}A_{22} = -\Delta_1^{-1}A_{12}.$$

In this case, we can compute X using $U_1 = TU_2$ (and without actually computing P).

When A is of order 2×2 and $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, the solution to $SDLCP(Q,\ L_A)$ is given by

$$X = \frac{(1 - \theta^2)}{a_{11}\theta^2 + a_{12}a_{21}\theta + a_{22}}\begin{bmatrix} \theta^2 & \theta \\ \theta & 1 \end{bmatrix},$$

where

$$\theta = \frac{-(a_{11} + a_{22}) + \sqrt{(a_{11} + a_{22})^2 - 4a_{12}a_{21}}}{2a_{21}}.$$

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References

[1] A. Albert, Conditions for positive and negative definiteness in terms of pseudo inverses, *SIAM Journal of Applied Mathematics* 17 (1969) 434-440 .

[2] R.W. Cottle, J.S. Pang and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, Inc., Boston, 1992.

[3] Ferris, M. C., Pang, J. S. : Engineering and economic applications of complementarity problems, SIAM Rev. 39 (1997) 669-713.

[4] Ferris, M.C., Pang, J. S., eds. : Complementarity and Variational Problems (State of the Art), Society for Industrial Applied Mathematics, Philadephia, 1997.

[5] Gowda, M. S., and T. Parthasarathy Complementarity forms of theorems of Lyaponov and Stein, and related results,Gowda, M. S., and Parthasarathy, T. . Complementarity forms of theorems of Lyaponov and Stein, and related results, Linear Algebra Appl., 320 (2000) 131-144.

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[6] M. S. Gowda and Y. Song, Semidefinite linear complementarity problems and a theorem of Lyapunov, Research Report, Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, Maryland, USA, March 1999.

[7] Karamardian, S.: An existence theorem for the complementarity problem. J. Optimization Theory Appl. 19 (1976) 227-232.

[8] Kojima, M., Shindoh, H., Hara, S. . Interior methods for the monotone semidefinite linear complementrity problems, SIAM J. Optim. 7 (1997) 86-125.

[9] Murty, K. G. . *Linear Complementarity, Linear And Nonlinear Programming*. Helderman Verlag, Berlin, 1988.

[10] Ouellette, D. V. . Schur complements and statistics, *Linear Algebra Appl.,* 36 (1981) 187-295.

[11] A. R. Rao and P. Bhimasankaram . *Linear Algebra*. Second Edition. Hindustan Book Agency (India), New Delhi, 2000.

[12] C. R. Rao . *Linear Statistical Inference and its Applications*. Second Edition. Wiley Eastern Private Limited, New Delhi , 1973.

[13] Samulson, H., Tharall, R. M., and Wesler, O. A partition theorem for Euclidean n-space. Proceedings of the American Mathematical Society 9 (1958) 805-807.

[14] Styan, G. P. H. Schur complements and linear statistical models. *Proceedings of First Tampere Seminar on Linear Models,* (1985) 37-75.