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Résumé de l'article

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Sensitivity analysis in convex quadratic optimization: Simultaneous perturbation of the objective and right-hand-side vectors

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Abstract

In this paper we study the behavior of Convex Quadratic Optimization problems when variation occurs simultaneously in the right-hand side vector of the constraints and in the coefficient vector of the linear term in the objective function. It is proven that the optimal value function is piecewise-quadratic. The concepts of transition point and invariancy interval are generalized to the case of simultaneous perturbation. Criteria for convexity, concavity or linearity of the optimal value function on invariancy intervals are derived. Furthermore, differentiability of the optimal value function is studied, and linear optimization problems are given to calculate the left and right derivatives. An algorithm, that is capable to compute the transition points and optimal partitions on all invariancy intervals, is outlined. We specialize the method to Linear Optimization problems and provide a practical example of simultaneous perturbation parametric quadratic optimization problem from electrical engineering.

Key words: Programming, quadratic: simultaneous perturbation sensitivity analysis using IPMs. Programming, linear, parametric: simultaneous perturbation.

1. Introduction

In this paper we are concerned with the sensitivity analysis of perturbed Convex Quadratic Optimization (CQO) problems where the coefficient vector of the linear term of the objective function and the right-hand side (RHS) vector of the constraints are varied simultaneously. This type of sensitivity analysis is often referred to as parametric programming. Research on the topic was triggered when a variant of parametric CQO problems was considered by Markowitz (1956). He developed the critical line method to determine the optimal value function of his parametric problem and applied it to mean-variance portfolio analysis. The basic result for parametric quadratic programming obtained by Markowitz is that the optimal value function (efficient frontier in financial terminology) is piecewise quadratic and can be obtained by computing successive corner portfolios, while, in between these corner portfolios, the optimal solutions vary linearly. Non-degeneracy was assumed and a variant of the simplex method was used for computations.

Difficulties that may occur in parametric analysis when the problem is degenerate are studied extensively in the Linear Optimization (LO) literature. In case of degeneracy the optimal basis need not be unique and multiple optimal solutions may exist. While simplex methods were used to perform the computations in earlier studies (see e.g., Murty (1983) for a comprehensive survey), recently research on parametric analysis was revisited from the point of view of interior-point methods (IPMs). For degenerate LO problems, the availability of strictly complementary solutions produced by IPMs allows to overcome many difficulties associated with the use of bases. Adler and Monteiro (1992) pioneered the use of IPMs in parametric analysis for LO (see also Jansen et al. (1997)). Berkelaar, Roos and Terlaky (1997) emphasized shortcomings of using optimal bases in parametric LO showing by an example that different optimal bases computed by different LO packages give different optimality intervals.

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Naturally, results obtained for parametric LO were extended to CQO. Berkelaar et al. (1997) showed that the optimal partition approach can be generalized to the quadratic case by introducing tripartition of variables instead of bipartition. They performed sensitivity analysis for the cases when perturbation occurs either in the coefficient vector of the linear term of the objective function or in the RHS of the constraints. In this paper we show that the results obtained in Berkelaar, Roos and Terlaky (1997) and Berkelaar et al. (1997) can be generalized further to accommodate simultaneous perturbation of the data even in the presence of degeneracy.

Considering simultaneous perturbation provides a unified approach to parametric LO and CQO problems that includes perturbation of the linear term coefficients in the objective function or the RHS vector of the constraints as its subcases. This approach makes the implementation easier and simplifies the explanation of the methodology. Theoretical results allow us to present a universal computational algorithm for the parametric analysis of LO/CQO problems. In many parametric models, simultaneous perturbation can be viewed as an underlying process where changes in the process influence the whole model. We describe such practical example of the adaptive power allocation between users of Digital Subscriber Lines (DSL).

Recall that CQO is a special case of Convex Conic Optimization (CCO). Recently, Yildirim (2004) has introduced an optimal partition concept for conic optimization. He took a pure geometric approach in defining the optimal partition while we use the algebraic one. Although, the geometric approach has the advantage of being independent from the representation of the underlying optimization problem, it has some deficiencies. The major difficulty is extracting the optimal partition from a high-dimensional geometric object and, consequently, it is inconvenient for numerical calculations. In contrast, the algebraic approach, used in this paper, is directly applicable for numerical implementation.

The principal novelty of our results is an algorithm that allows to identify all invariancy intervals iteratively and thus differs significantly from all the work done in simultaneous perturbation analysis so far.

The paper is organized as follows. In Section 2, the CQO problem is introduced and some elementary concepts are reviewed. Simple properties of the optimal value function are summarized in Section 3. Section 4 is devoted to deriving more properties of the optimal value function. It is shown that the optimal value function is continuous and piecewise quadratic, and an explicit formula is presented to identify it on the subintervals. Criteria for convexity, concavity or linearity of the optimal value function on these subintervals are derived. We investigate the first and second order derivatives of the optimal value function as well. Auxiliary LO problems can be used to compute the left and right derivatives. It is shown that the optimal partition on the neighboring intervals can be identified by solving an auxiliary self-dual CQO problem. The results are summarized in a computational algorithm for which implementation issues are discussed as well. Specialization of our method to LO problems is described in Section 5. For illustration, the results are tested on a simple problem in Section 6. A recent application of parametric CQO described in Section 7 arises from electrical engineering and it is based on recent developments in optimal multi-user spectrum management for Digital Subscriber Lines (DSL). We conclude the paper with some remarks and we sketch further research directions.

2. Preliminaries

A primal CQO problem is defined as:

$$(QP) \quad \min\,\{\, c^T x + \frac{1}{2} x^T Q x \;:\; Ax = b,\; x \geq 0 \,\}\,,$$

where $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix, $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are fixed data and $x \in \mathbb{R}^n$ is an unknown vector.

The Wolfe-Dual of (QP) is given by

$$(QD) \quad \max\,\{\, b^T y - \frac{1}{2} u^T Q u \;: \\ A^T y + s - Qu = c,\; s \geq 0 \,\}\,,$$

where $s, u \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are unknown vectors. The feasible regions of (QP) and (QD) are denoted by

$$\mathcal{QP} = \{x : Ax = b, x \geq 0\}\,,$$
$$\mathcal{QD} = \left\{(u, y, s) : A^T y + s - Qu = c,\; s, u \geq 0\right\}\,,$$

and their associated optimal solutions sets are $\mathcal{QP}^*$ and $\mathcal{QD}^*$, respectively. It is well known that for any optimal solution of (QP) and (QD) we have $Qx = Qu$ and $s^T x = 0$, see e.g., Dorn (1960). Having zero duality gap, i.e., $s^T x = 0$ is equivalent to $s_i x_i = 0$ for all $i \in \{1, 2, \ldots, n\}$. This property of the nonnegative variables x and s is called the *complementarity property*. It is obvious that there are optimal solutions with $x = u$. Since we are only interested in the solutions where $x = u$, u will henceforth be replaced by x in the dual problem. It is easy to show, see e.g., Berkelaar et al. (1997) and Dorn (1960), that for any two optimal solutions (x^*, y^*, s^*) and $(\tilde{x}, \tilde{y}, \tilde{s})$ of (QP) and (QD) it holds that $Qx^* = Q\tilde{x}$, $c^T x^* = c^T \tilde{x}$ and $b^T y^* = b^T \tilde{y}$ and consequently,

$$\tilde{x}^T s^* = \tilde{s}^T x^* = 0. \tag{1}$$

The *optimal partition* of the index set $\{1, 2, \ldots, n\}$ is defined as

$$\begin{aligned} \mathcal{B} &= \{\ i : x_i > 0 \text{ for an optimal solution } x \in \mathcal{QP}^*\}, \\ \mathcal{N} &= \{\ i : s_i > 0 \text{ for an optimal solution } \\ &\qquad (x, y, s) \in \mathcal{QD}^*\}, \\ \mathcal{T} &= \{1, 2, \ldots, n\} \setminus (\mathcal{B} \cup \mathcal{N}), \end{aligned}$$

and denoted by $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. Berkelaar et al. (1997) and Berkelaar, Roos and Terlaky (1997) showed that this partition is unique. The *support set* of a vector v is defined as $\sigma(v) = \{i : v_i > 0\}$ and is used extensively in this paper. An optimal solution (x, y, s) is called *maximally complementary* if it possesses the following properties:

$$\begin{aligned} x_i > 0 &\text{ if and only if } i \in \mathcal{B}, \\ s_i > 0 &\text{ if and only if } i \in \mathcal{N}. \end{aligned}$$

For any maximally complementary solution (x, y, s) the relations $\sigma(x) = \mathcal{B}$ and $\sigma(s) = \mathcal{N}$ hold. The existence of a maximally complementary solution is a direct consequence of the convexity of the optimal sets $\mathcal{QP}^*$ and $\mathcal{QD}^*$. It is known that IPMs find a maximally complementary solution in the limit, see e.g., McLinden (1980) and Güler and Ye (1993).

The general perturbed CQO problem is

$$(QP_{\lambda_b,\lambda_c}) \quad \min\{\, (c + \lambda_c \triangle c)^T x + \frac{1}{2} x^T Q x \ : \\ Ax = b + \lambda_b \triangle b,\ x \geq 0\,\},$$

where $\triangle b \in \mathbb{R}^m$ and $\triangle c \in \mathbb{R}^n$ are nonzero perturbation vectors, and λ_b and λ_c are real parameters. The optimal value function $\phi(\lambda_b, \lambda_c)$ denotes the optimal value of $(QP_{\lambda_b,\lambda_c})$ as the function of the parameters λ_b and λ_c. As we already mentioned, Berkelaar, Roos and Terlaky (1997) and Berkelaar et al. (1997) were the first to analyze parametric CQO by using the optimal partition approach when variation occurs either in the RHS or the linear term of the objective function data, i.e., either when $\triangle c$ or $\triangle b$ is zero. In these cases the domain of the optimal value function $\phi(\lambda_b, 0)$ (or $\phi(0, \lambda_c)$) is a closed interval of the real line and the function is piecewise convex (concave) quadratic on its domain. The authors presented an explicit formula for the optimal value function on these subintervals and introduced the concept of transition points that separate them. They proved that the optimal partition is invariant on the subintervals which are characterized by consecutive transition points. The authors also studied the behavior of first and second order derivatives of the optimal value function and proved that the transition points coincide with the points where first or second order derivatives do not exist. It was proven that by solving auxiliary self-dual CQO problems, one can identify the optimal partitions on the neighboring subintervals.

The results obtained by Yildirim (2004) for the simultaneous perturbation case in conic optimization and by using the geometric definition of the optimal partition may be linked to our findings. In his paper, Yildirim introduced the concept of the invariancy interval and presented auxiliary problems to identify the lower and upper bounds of the invariancy interval that contains the given parameter value. He also proved that the optimal value function is quadratic on the current invariancy interval. Although Yildirim's results are very interesting in the light of extending parametric optimization techniques to conic optimization problems, there are some obstacles that prevent direct mapping of them to our methodology as we will explain in Section 4. Generally speaking, the optimal partition is well-defined if the primal and dual conic optimization problems have nonempty optimal solution sets and the duality gap is zero. Due to the more general setting Yildirim (2004) presents optimization problems defined on relative interiors of primal and dual sets. Those open set formulations (relative interiors of feasible sets, see e.g. problem (9) in Yildirim's paper) are less appropriate to direct calculations than the standard form problems defined in this paper.

3. The Optimal Value Function in Simultaneous Perturbation Sensitivity Analysis

In this section, we introduce explicitly the perturbed CQO problem when perturbation simultaneously occurs in the RHS data and the linear term of the objective value function of (QP). In the problem $(QP_{\lambda_b,\lambda_c})$ that was introduced in the pervious section, λ_b and λ_c are independent parameters. In this paper we are only concerned with the case when they coincide, i.e., when $\lambda_b = \lambda_c = \lambda$. Consequently, the perturbation takes the form λh, where $h = (\triangle b^T, \triangle c^T)^T \in \mathbb{R}^{m+n}$ is a nonzero perturbing direction and $\lambda \in \mathbb{R}$ is a parameter. Thus, we define the following primal and dual perturbed problems corresponding to (QP) and (QD), respectively:

$$(QP_\lambda) \quad \min \{ (c + \lambda \triangle c)^T x + \frac{1}{2} x^T Q x : \; Ax = b + \lambda \triangle b, \; x \geq 0 \},$$

$$(QD_\lambda) \quad \max \{ (b + \lambda \triangle b)^T y - \frac{1}{2} x^T Q x : \; A^T y + s - Qx = c + \lambda \triangle c, \; s \geq 0 \}.$$

The solution methodology for the problem (QP_λ) is our primary interest here. Let $\mathcal{QP}_\lambda$ and $\mathcal{QD}_\lambda$ denote the feasible sets of the problems (QP_λ) and (QD_λ), respectively. Their optimal solution sets are analogously denoted by $\mathcal{QP}^*_\lambda$ and $\mathcal{QD}^*_\lambda$. The optimal value function of (QP_λ) and (QD_λ) is

$$\begin{aligned}\phi(\lambda) &= (c + \lambda \triangle c)^T x^*(\lambda) + \frac{1}{2} x^*(\lambda)^T Q x^*(\lambda) \\ &= (b + \lambda \triangle b)^T y^*(\lambda) - \frac{1}{2} x^*(\lambda)^T Q x^*(\lambda),\end{aligned}$$

where $x^*(\lambda) \in \mathcal{QP}^*_\lambda$ and $(x^*(\lambda), y^*(\lambda), s^*(\lambda)) \in \mathcal{QD}^*_\lambda$. Further, we define

$$\phi(\lambda) = +\infty \;\text{ if } \mathcal{QP}_\lambda = \emptyset,$$

$$\phi(\lambda) = -\infty \;\text{ if } \mathcal{QP}_\lambda \neq \emptyset \text{ and } (QP_\lambda) \text{ is unbounded.}$$

Let us denote the domain of $\phi(\lambda)$ by

$$\Lambda = \{\lambda : \mathcal{QP}_\lambda \neq \emptyset \text{ and } \mathcal{QD}_\lambda \neq \emptyset\}.$$

Since it is assumed that (QP) and (QD) have optimal solutions, it follows that $\Lambda \neq \emptyset$. We can easily prove the following property of Λ.

Lemma 1 $\Lambda \subseteq \mathbb{R}$ *is a closed interval.*

Proof: First, we prove that the set Λ is connected and so it is an interval of the real line. Let $\lambda_1, \lambda_2 \in \Lambda$ be two arbitrary numbers. Let $(x(\lambda_1), y(\lambda_1), s(\lambda_1)) \in \mathcal{QP}_{\lambda_1} \times \mathcal{QD}_{\lambda_1}$ and $(x(\lambda_2), y(\lambda_2), s(\lambda_2)) \in \mathcal{QP}_{\lambda_2} \times \mathcal{QD}_{\lambda_2}$ be known. For any $\lambda \in (\lambda_1, \lambda_2)$ and $\theta = \frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1}$ we have

$$\lambda = \theta \lambda_1 + (1 - \theta) \lambda_2.$$

Let us define

$$\begin{aligned}x(\lambda) &= \theta x(\lambda_1) + (1 - \theta) x(\lambda_2), \\ y(\lambda) &= \theta y(\lambda_1) + (1 - \theta) y(\lambda_2), \\ s(\lambda) &= \theta s(\lambda_1) + (1 - \theta) s(\lambda_2).\end{aligned}$$

By construction $(x(\lambda), y(\lambda), s(\lambda)) \in \mathcal{QP}_\lambda \times \mathcal{QD}_\lambda$, thus $\mathcal{QP}_\lambda \neq \emptyset$ and $\mathcal{QD}_\lambda \neq \emptyset$. This implies the set Λ is connected.

Second, we prove the closedness of Λ. Let $\lambda \notin \Lambda$. There are two cases: the primal problem (QP_λ) is feasible but unbounded or it is infeasible. We only prove the second case, the first one can be proved analogously. If the primal problem (QP_λ) is infeasible then by the Farkas Lemma (see e.g., Murty (1983) or Roos, Terlaky and Vial (2006)) there is a vector y such that $A^T y \leq 0$ and $(b + \lambda \triangle b)^T y > 0$. Fixing y and considering λ as a variable, the set $S(y) = \left\{\lambda : (b + \lambda \triangle b)^T y > 0\right\}$ is an open half-line in λ, thus the given vector y is a certificate of infeasibility of (QP_λ) for an open interval. Thus, the union $\bigcup_y S(y)$, where y is a Farkas certificate for the infeasibility of (QP_λ) for some $\lambda \in \mathbb{R}$, is open. Consequently, the domain of the optimal value function is closed. The proof is complete.

4. Properties of the Optimal Value Function

In this section we investigate the properties of the optimal value function. These are generalizations of the corresponding properties that have been proven in Berkelaar et al. (1997) for the case when $\triangle c = 0$ or $\triangle b = 0$. We also explain the relation of our results in Section 4.1 to the ones obtained by Yildirim (2004). In contrast, Sections 4.2 and 4.3 contain the new results with respect to simultaneous perturbations in the CQO case.

4.1. *Basic Properties*

For $\lambda^* \in \Lambda$, let $\pi = \pi(\lambda^*)$ denote the optimal partition and let (x^*, y^*, s^*) be a maximally complementary solution at λ^*. We use the following notation that generalizes the notation introduced in Berkelaar et al. (1997):

$$\begin{aligned}
\mathcal{O}(\pi) &= \{\lambda \in \Lambda : \pi(\lambda) = \pi\}; \\
\mathcal{S}_\lambda(\pi) &= \{(x, y, s) : x \in \mathcal{QP}_\lambda, \; (x, y, s) \in \mathcal{QD}_\lambda, \\
&\qquad x_\mathcal{B} > 0, \; x_{\mathcal{N} \cup \mathcal{T}} = 0, \; s_\mathcal{N} > 0, \; s_{\mathcal{B} \cup \mathcal{T}} = 0\}; \\
\overline{\mathcal{S}}_\lambda(\pi) &= \{(x, y, s) : x \in \mathcal{QP}_\lambda, (x, y, s) \in \mathcal{QD}_\lambda, \\
&\qquad x_\mathcal{B} \geq 0, \; x_{\mathcal{N} \cup \mathcal{T}} = 0, \; s_\mathcal{N} \geq 0, \; s_{\mathcal{B} \cup \mathcal{T}} = 0\}; \\
\Lambda(\pi) &= \{\lambda \in \Lambda : \mathcal{S}_\lambda(\pi) \neq \emptyset\}; \\
\overline{\Lambda}(\pi) &= \left\{\lambda \in \Lambda : \overline{\mathcal{S}}_\lambda(\pi) \neq \emptyset\right\}; \\
D_\pi &= \{(\triangle x, \triangle y, \triangle s) : A \triangle x = \triangle b, \\
&\qquad A^T \triangle y + \triangle s - Q \triangle x = \triangle c, \triangle x_{\mathcal{N} \cup \mathcal{T}} = 0, \\
&\qquad \triangle s_{\mathcal{B} \cup \mathcal{T}} = 0\}.
\end{aligned}$$

Here $\mathcal{O}(\pi)$ denotes a set of parameter values for which the optimal partition π is constant. Further, $\mathcal{S}_\lambda(\pi)$ is the primal-dual optimal solution set of maximally complementary optimal solutions of the perturbed primal and dual CQO problems for the parameter value $\lambda \in \mathcal{O}(\pi)$. $\Lambda(\pi)$ denotes the set of parameter values for which the perturbed primal and dual problems have an optimal solution (x, y, s) such that $\sigma(x) = \mathcal{B}$ and $\sigma(s) = \mathcal{N}$. Besides, D_π refers to the variation vector of the primal-dual optimal solution of the perturbed CQO problem for any $\lambda \in \mathcal{O}(\pi)$, when the vectors $\triangle b$ and $\triangle c$ are given. Finally, $\overline{S}_\lambda(\pi)$ is the closure of $S_\lambda(\pi)$ for all $\lambda \in \Lambda(\pi)$ and $\overline{\Lambda}(\pi)$ is the closure of $\Lambda(\pi)$.

The following theorem resembles Theorem 3.1 from Berkelaar et al. (1997) and presents the basic relations between the open interval where the optimal partition is invariant and its closure. The proof can be found in the Appendix.

Theorem 2 *Let $\pi = \pi(\lambda^*) = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ denote the optimal partition for some λ^* and (x^*, y^*, s^*) denote an associated maximally complementary solution at λ^*. Then,*

(i) $\Lambda(\pi) = \{\lambda^*\}$ if and only if $D_\pi = \emptyset$;

(ii) $\Lambda(\pi)$ is an open interval if and only if $D_\pi \neq \emptyset$;

(iii) $\mathcal{O}(\pi) = \Lambda(\pi)$ and $cl\, \mathcal{O}(\pi) = cl\, \Lambda(\pi) = \overline{\Lambda}(\pi)$;

(iv) $\overline{\mathcal{S}}_\lambda(\pi) = \{(x, y, s) : x \in \mathcal{QP}^*_\lambda, (x, y, s) \in \mathcal{QD}^*_\lambda\}$ for all $\lambda \in \Lambda(\pi)$.

The following two corollaries are direct consequences of Theorem 2.

Corollary 3 *Let $\lambda_2 > \lambda_1$ be such that $\pi(\lambda_1) = \pi(\lambda_2)$. Then, $\pi(\lambda)$ is constant for all $\lambda \in [\lambda_1, \lambda_2]$.*

Corollary 4 *Let $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ be maximally complementary solutions of (QP_{λ_1}), (QD_{λ_1}) and (QP_{λ_2}), (QD_{λ_2}), respectively. Furthermore, let $(x(\lambda), y(\lambda), s(\lambda))$ be defined as*

$$x(\lambda) = \frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1} x^{(1)} + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} x^{(2)},$$
$$y(\lambda) = \frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1} y^{(1)} + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} y^{(2)},$$
$$s(\lambda) = \frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1} s^{(1)} + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} s^{(2)},$$

for any $\lambda \in [\lambda_1, \lambda_2]$. If there is an optimal partition π such that $\lambda_1, \lambda_2 \in \Lambda(\pi)$, then $(x(\lambda), y(\lambda), s(\lambda))$ is a maximally complementary solution of (QP_λ) and (QD_λ). Moreover, if $(x(\lambda), y(\lambda), s(\lambda))$ is a maximally complementary optimal solution, then $\lambda_1, \lambda_2 \in \overline{\Lambda}(\pi)$.

Proof: The first part of the statement is a direct consequence of the convexity of the optimal solution set.

Let us assume that $(x(\lambda), y(\lambda), s(\lambda))$ is a maximally complementary optimal solution with the optimal partition $\pi(\lambda) = (\mathcal{B}(\lambda), \mathcal{N}(\lambda), \mathcal{T}(\lambda))$. In this case, $\sigma(x^{(1)}) \subseteq \mathcal{B}(\lambda)$ and $\sigma(x^{(2)}) \subseteq \mathcal{B}(\lambda)$. Moreover, $\sigma(s^{(1)}) \subseteq \mathcal{N}(\lambda)$ and $\sigma(s^{(2)}) \subseteq \mathcal{N}(\lambda)$. Let us restrict our analysis to the characteristics of the optimal solution $(x^{(1)}, y^{(1)}, s^{(1)})$. Analogous reasoning applies for $(x^{(2)}, y^{(2)}, s^{(2)})$. We might have three cases for the support sets of $x^{(1)}$ and $s^{(1)}$.

- Case 1: $\sigma(x^{(1)}) = \mathcal{B}(\lambda)$ and $\sigma(s^{(1)}) = \mathcal{N}(\lambda)$.
- Case 2: $\sigma(x^{(1)}) = \mathcal{B}(\lambda)$ and $\sigma(s^{(1)}) \subset \mathcal{N}(\lambda)$.
- Case 3: $\sigma(x^{(1)}) \subset \mathcal{B}(\lambda)$ and $\sigma(s^{(1)}) = \mathcal{N}(\lambda)$.

For Case 1, it is obvious that $\lambda_1 \in \Lambda(\pi(\lambda))$ and the statement is valid. We prove that in Cases 2 and 3, λ_1 is one of the end points of $\overline{\Lambda}(\pi(\lambda))$. To the contrary, let λ_1 be not one of the end points of $\overline{\Lambda}(\pi(\lambda))$. It is clear that $\lambda_1 \notin \Lambda(\pi(\lambda))$. Without loss of generality, let us assume that it belongs to the immediate (open) interval to the right of $\overline{\Lambda}(\pi(\lambda))$. Thus, some convex combination of (x^*, s^*, y^*) and $(x^{(1)}, s^{(1)}, y^{(1)})$ lie outside of $\Lambda(\pi(\lambda))$ but with the same optimal partition $\pi(\lambda)$ that contradicts the definition of $\Lambda(\pi(\lambda))$. Thus, λ_1 is one of the end points of $\overline{\Lambda}(\pi(\lambda))$ and the proof is complete.

Though Yildirim (2004) has stated a theorem to identify the invariancy interval for general conic optimization problems, the direct specialization of his results to the CQO case is not straightforward. It is easily seen that having open sets in his Theorem 4.1, reduces to a closed formulation that we present in the following theorem. However, the major obstacle of efficient use of his method came back to the fact that correctly identifying the optimal partition from an approximate optimal solution is almost impossible. Here, we provide the following theorem that is based on standard equations and inequalities imposed on variables x and s. Consequently, it allows us to compute the endpoints of the interval $\overline{\Lambda}(\pi)$ efficiently. The proof of Theorem 5 is similar to the one of Theorem 48 of Berkelaar, Roos and Terlaky (1997).

Theorem 5 *Let $\lambda^* \in \Lambda$ and let (x^*, y^*, s^*) be a maximally complementary solution of (QP_{λ^*}) and (QD_{λ^*}) with optimal partition $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. Then the left and right extreme points of the closed interval $\overline{\Lambda}(\pi) = [\lambda_\ell, \lambda_u]$ that contains λ^* can be obtained by minimizing and maximizing λ over $\overline{\mathcal{S}}_\lambda(\pi)$, respectively, i.e., by* *solving*

$$\lambda_\ell = \min_{\lambda,x,y,s} \{ \lambda : Ax - \lambda\triangle b = b, \quad x_{\mathcal{B}} \geq 0, x_{\mathcal{N}\cup\mathcal{T}} = 0, \quad A^T y + s - Qx - \lambda\triangle c = c, \quad s_{\mathcal{N}} \geq 0, s_{\mathcal{B}\cup\mathcal{T}} = 0 \}, \tag{2}$$

and

$$\lambda_u = \max_{\lambda,x,y,s} \{ \lambda : Ax - \lambda\triangle b = b, \quad x_{\mathcal{B}} \geq 0,\ x_{\mathcal{N}\cup\mathcal{T}} = 0, \quad A^T y + s - Qx - \lambda\triangle c = c, \quad s_{\mathcal{N}} \geq 0,\ s_{\mathcal{B}\cup\mathcal{T}} = 0 \}. \tag{3}$$

The open interval $\Lambda(\pi)$ is referred to as *invariancy interval* because the optimal partition is invariant on it. The points λ_ℓ and λ_u, that separate neighboring invariancy intervals, are called *transition points*.

Remark 6 *Note that π represents either an optimal partition at a transition point, when $\lambda_\ell = \lambda_u$, or on the interval between two consequent transition points λ_ℓ and λ_u. Thus $\Lambda = \bigcup_\pi \Lambda(\pi) = \bigcup_\pi \overline{\Lambda}(\pi)$, where π runs throughout all possible partitions.*

It is worth mentioning that Yildirim (2004) proved that the optimal value function is quadratic on any invariancy interval, and presented an example showing that this function might be neither convex nor concave. His proof is based on a strictly complementary solution computed for the current parameter value $\lambda = 0$. Here, we present an explicit representation of the optimal value function on an invariancy interval by utilizing primal-dual optimal solutions (not necessarily maximally complementary) for two arbitrarily chosen parameter values inside this interval. We also provide simple criteria to determine the convexity, concavity or linearity of the optimal value function on an invariancy interval. We start with the following theorem.

Theorem 7 *Let $\lambda_\ell < \lambda_u$ be obtained by solving (2) and (3), respectively. The optimal value function $\phi(\lambda)$ is quadratic on $\mathcal{O}(\pi) = (\lambda_\ell, \lambda_u)$.*

Proof: Let $\lambda_\ell < \lambda_1 < \lambda < \lambda_2 < \lambda_u$ be given and let $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ be pairs of primal-dual optimal solutions corresponding to λ_1 and λ_2, respectively. So, using $\theta = \frac{\lambda-\lambda_1}{\lambda_2-\lambda_1} \in (0,1)$ allows us to give an explicit expression for the optimal solution $(x(\lambda), y(\lambda), s(\lambda))$ as

$$x(\lambda) = x^{(1)} + \theta\triangle x,$$
$$y(\lambda) = y^{(1)} + \theta\triangle y,$$
$$s(\lambda) = s^{(1)} + \theta\triangle s,$$

where $\triangle x = x^{(2)} - x^{(1)}$, $\triangle y = y^{(2)} - y^{(1)}$, $\triangle s = s^{(2)} - s^{(1)}$ and $(x(\lambda), y(\lambda), s(\lambda))$ is a pair of primal-dual optimal solution corresponding to λ. Denoting $\triangle\lambda = \lambda_2 - \lambda_1$, we also have

$$A\triangle x = \triangle\lambda\triangle b, \tag{4}$$
$$A^T\triangle y + \triangle s - Q\triangle x = \triangle\lambda\triangle c. \tag{5}$$

The optimal value function at λ is given by

$$\begin{aligned}\phi(\lambda) &= (b + \lambda\triangle b)^T y(\lambda) - \tfrac{1}{2}x(\lambda)^T Q x(\lambda)\\ &= (b + (\lambda_1 + \theta\triangle\lambda)\triangle b)^T (y^{(1)} + \theta\triangle y)\\ &\quad -\tfrac{1}{2}(x^{(1)} + \theta\triangle x)^T Q (x^{(1)} + \theta\triangle x)\\ &= (b + \lambda_1\triangle b)^T y^{(1)} + \theta(\triangle\lambda\triangle b^T y^{(1)}\\ &\quad +(b + \lambda_1\triangle b)^T\triangle y) + \theta^2\triangle\lambda\triangle b^T\triangle y\\ &\quad -\tfrac{1}{2}{x^{(1)}}^T Q x^{(1)} - \theta {x^{(1)}}^T Q\triangle x\\ &\quad -\tfrac{1}{2}\theta^2\triangle x^T Q\triangle x.\end{aligned} \tag{6}$$

From equations (4) and (5), one gets

$$\triangle x^T Q\triangle x = \triangle\lambda(\triangle b^T\triangle y - \triangle c^T\triangle x), \tag{7}$$
$${x^{(1)}}^T Q\triangle x = (b + \lambda_1\triangle b)^T\triangle y - \triangle\lambda\triangle c^T x^{(1)}. \tag{8}$$

Substituting (7) and (8) into (6) we obtain

$$\begin{aligned}\phi(\lambda) &= \phi(\lambda_1 + \theta\triangle\lambda)\\ &= \phi(\lambda_1) + \theta\triangle\lambda(\triangle b^T y^{(1)} + \triangle c^T x^{(1)})\\ &\quad + \frac{1}{2}\theta^2\triangle\lambda(\triangle c^T\triangle x + \triangle b^T\triangle y).\end{aligned} \tag{9}$$

Using the notation

$$\gamma_1 = \triangle b^T y^{(1)} + \triangle c^T x^{(1)}, \tag{10}$$
$$\gamma_2 = \triangle b^T y^{(2)} + \triangle c^T x^{(2)}, \tag{11}$$
$$\gamma = \frac{\gamma_2 - \gamma_1}{\lambda_2 - \lambda_1} = \frac{\triangle c^T\triangle x + \triangle b^T\triangle y}{\lambda_2 - \lambda_1}, \tag{12}$$

one can rewrite (9) as

$$\phi(\lambda) = (\phi(\lambda_1) - \lambda_1\gamma_1 + \frac{1}{2}\lambda_1^2\gamma) + (\gamma_1 - \lambda_1\gamma)\lambda + \frac{1}{2}\gamma\lambda^2. \tag{13}$$

Because λ_1 and λ_2 are two arbitrary elements from the interval $(\lambda_\ell, \lambda_u)$, the claim of the theorem follows directly from (13). The proof is complete.

It should be mentioned that the sign of $\triangle c^T \triangle x + \triangle b^T \triangle y$ in (9) is independent of λ_1 and λ_2, because both λ_1 and λ_2 are two arbitrary numbers in $(\lambda_\ell, \lambda_u)$. The following corollary is a straightforward consequence of (13).

Corollary 8 *For two arbitrary $\lambda_1 < \lambda_2 \in (\lambda_\ell, \lambda_u)$, let $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ be pairs of primal-dual optimal solutions corresponding to λ_1 and λ_2, respectively. Moreover, let $\triangle x = x^{(2)} - x^{(1)}$ and $\triangle y = y^{(2)} - y^{(1)}$. Then, the optimal value function $\phi(\lambda)$ is quadratic on $\mathcal{O}(\pi) = (\lambda_\ell, \lambda_u)$ and it is*

(i) strictly convex if $\triangle c^T \triangle x + \triangle b^T \triangle y > 0$;
(ii) linear if $\triangle c^T \triangle x + \triangle b^T \triangle y = 0$;
(iii) strictly concave if $\triangle c^T \triangle x + \triangle b^T \triangle y < 0$.

Corollary 9 *The optimal value function $\phi(\lambda)$ is continuous and piecewise quadratic on Λ.*

Proof: The fact that the optimal value function is piecewise quadratic follows directly from Theorem 7. Recall that the feasible solution sets of problems (2) and (3) are closed convex sets and for any $\lambda \in (\lambda_\ell, \lambda_u)$ there is a corresponding vector $(x(\lambda), y(\lambda), s(\lambda))$ that is an optimal solution of the perturbed problems (QP_λ) and (QD_λ). Consider problem (2) and pick any sequence converging to an optimal solution of (2). Relying on the fact that any feasible solution corresponding to a $\lambda \in (\lambda_\ell, \lambda_u)$ is an optimal solution of (QP_λ) and (QD_λ), it follows that the optimal value function is continuous.

Two auxiliary LO problems were presented in Theorem 5 to identify transition points and consequently to determine invariancy intervals. A logical question that appears here is how to proceed from the initial invariancy interval to a neighboring one iteratively to cover the whole domain of λ. It turns out that we need to compute the derivatives of the optimal value function for that. It is done as described in the following theorem that is the specialization of Corollary 5.1 in Yildirim (2004) to quadratic problems.

Theorem 10 *For a given $\lambda \in \Lambda$, the left and right derivatives of the optimal value function $\phi(\lambda)$ at λ satisfy*

$$\phi'_-(\lambda) = \min_{x,y,s}\{\triangle b^T y : (x,y,s) \in \mathcal{QD}^*_\lambda\} + \max_x\{\triangle c^T x : x \in \mathcal{QP}^*_\lambda\}, \tag{14}$$

$$\phi'_+(\lambda) = \max_{x,y,s}\{\triangle b^T y : (x,y,s) \in \mathcal{QD}^*_\lambda\} + \min_x\{\triangle c^T x : x \in \mathcal{QP}^*_\lambda\}. \tag{15}$$

Remark 11 *If λ is not a transition point, then the optimal value function at λ is a differentiable quadratic function and its first order derivative is*

$$\phi'(\lambda) = \triangle b^T y(\lambda) + \triangle c^T x(\lambda).$$

Here, $(x(\lambda), y(\lambda), s(\lambda))$ is any pair of primal-dual optimal solution corresponding to λ.

4.2. *Relation between Derivatives, Invariancy Intervals, and Transition Points*

In this subsection, we use basic properties of the optimal value function and its derivatives to investigate the relationship between the invariancy intervals and neighboring transition points where these derivatives may not exist. We also show how we can proceed from one invariancy interval to another to cover the whole interval Λ. These results allow us to develop our algorithm for solving parametric CQO problems.

It is worthwhile to make some remarks about Theorem 10 first. It seems that we need to solve two optimization problems to find the right or left first-order derivatives of the optimal value function at a transition point. Actually we can combine these two problems into one. We consider problem (15) only. Similar results hold for problem (14). Let (x^*, y^*, s^*) be a pair of primal-dual optimal solutions of (QP_λ) and (QD_λ) and

$$\begin{aligned}\mathcal{QPD}^*_\lambda = \{(x,y,s) :\ & Ax = b + \lambda\triangle b,\\ & x \geq 0,\ x^T s^* = 0,\ Qx = Qx^*,\\ & A^T y + s - Qx = c + \lambda\triangle c,\\ & s \geq 0,\ s^T x^* = 0\}.\end{aligned}$$

First, in the definition of the set $\mathcal{QPD}^*_\lambda$ the constraints $x \geq 0,\ x^T s^* = 0,\ Qx = Qx^*$ and $s \geq 0,\ s^T x^* = 0$ are equivalent to $x_\mathcal{B} \geq 0,\ x_{\mathcal{N}\cup\mathcal{T}} = 0$ and $s_\mathcal{N} \geq 0,\ s_{\mathcal{B}\cup\mathcal{T}} = 0$, where $(\mathcal{B}, \mathcal{N}, \mathcal{T})$ is the optimal partition at the transition point λ. The fact that $x_\mathcal{B} \geq 0$ directly follows from $x \geq 0$. On the other hand, since (x,y,s) is a primal-dual optimal solution and (x^*, y^*, s^*) is a maximally complementary optimal solution, then $\sigma(x) \subseteq \sigma(x^*)$, thus $x_{\mathcal{N}\cup\mathcal{T}} = 0$ is its immediate result. Analogous reasoning is valid for $s_{\mathcal{B}\cup\mathcal{T}} = 0$. Second, let us consider the first and the second subproblems of (15). Observe that the optimal solutions produced by each subproblem are both optimal for (QP_λ) and (QD_λ) and so the vector Qx, appearing in the constraints, is always identical for both subproblems (see, e.g., Dorn 1960). This means that we can maximize the first subproblem over $\mathcal{QPD}^*_\lambda$ and minimize the second subproblem over $\mathcal{QPD}^*_\lambda$ simultaneously. In other words, instead of solving two subproblems in (15) separately, we can solve the problem

$$\min_{x,y,s}\{\triangle c^T x - \triangle b^T y : (x,y,s) \in \mathcal{QPD}^*_\lambda\} \qquad (16)$$

that produces the same optimal solution $(\hat{x},\hat{y},\hat{s})$ as a solution of problem (15). Then the right derivative $\phi'_+(\lambda)$ can be computed by using the values $(\hat{x},\hat{y},\hat{s})$ as $\phi'_+(\lambda) = \triangle b^T\hat{y} + \triangle c^T\hat{x}$. Consequently, we refer to the optimal solutions of problems (15) and (16) interchangeably.

The next lemma shows an important property of strictly complementary solutions of (14) and (15) and will be used later on in the paper.

Lemma 12 *Let λ^* be a transition point of the optimal value function. Further, assume that the (open) invariancy interval to the right of λ^* contains $\overline{\lambda}$ with the optimal partition $\overline{\pi} = (\overline{\mathcal{B}},\overline{\mathcal{N}},\overline{\mathcal{T}})$. Let (x,y,s) be an optimal solution of (15) with $\lambda = \lambda^*$. Then, $\sigma(x) \subseteq \overline{\mathcal{B}}$ and $\sigma(s) \subseteq \overline{\mathcal{N}}$.*

Proof: Let $(\overline{x},\overline{y},\overline{s})$ be a maximally complementary solution at $\overline{\lambda}$ and let $(\lambda^*,\underline{x},\underline{y},\underline{s})$ be an optimal solution of (2) obtained for the optimal partition $\pi = \overline{\pi}$.

First, we want to prove that

$$\triangle c^T x = \triangle c^T \underline{x} \quad \text{and} \quad \triangle b^T y = \triangle b^T \underline{y}, \qquad (17)$$

$$c^T x = c^T \underline{x} \quad \text{and} \quad b^T y = b^T \underline{y}. \qquad (18)$$

For this purpose we use equation (9). In (9) and (10-12) let $\lambda_2 = \overline{\lambda}$, $x^{(2)} = \overline{x}$, $y^{(2)} = \overline{y}$. Continuity of the optimal value function, that is proved in Corollary 9, allows us to establish that equation (9) holds not only on invariancy intervals, but also at their endpoints, i.e., at the transition points. Thus, we are allowed to consider the case when $\lambda_1 = \lambda^*$ and $(x^{(1)},y^{(1)},s^{(1)})$ is any optimal solution at the transition point λ^*.

Computing $\phi(\lambda)$ at the point $\overline{\lambda}$ (where $\theta = \frac{\lambda-\lambda_1}{\lambda_2-\lambda_1} = \frac{\overline{\lambda}-\lambda^*}{\overline{\lambda}-\lambda^*} = 1$) by (9) gives us

$$\begin{aligned}\phi(\overline{\lambda}) &= \phi(\lambda^*) + (\overline{\lambda}-\lambda^*)(\triangle b^T y^{(1)} + \triangle c^T x^{(1)}) \\ &\quad + \frac{1}{2}(\overline{\lambda}-\lambda^*)[\triangle c^T(\overline{x}-x^{(1)}) \\ &\quad + \triangle b^T(\overline{y}-y^{(1)})] \\ &= \phi(\lambda^*) + \frac{1}{2}(\overline{\lambda}-\lambda^*)[\triangle c^T(\overline{x}+x^{(1)}) \\ &\quad + \triangle b^T(\overline{y}+y^{(1)})]. \end{aligned} \qquad (19)$$

One can rearrange (19) as

$$\frac{\phi(\overline{\lambda})-\phi(\lambda^*)}{\overline{\lambda}-\lambda^*} = \triangle c^T\left(\frac{\overline{x}+x^{(1)}}{2}\right) + \triangle b^T\left(\frac{\overline{y}+y^{(1)}}{2}\right).$$

Let $\overline{\lambda} \downarrow \lambda^*$, then we have

$$\begin{aligned}\phi'_+(\lambda^*) &= \lim_{\overline{\lambda}\downarrow\lambda^*}\frac{\phi(\overline{\lambda})-\phi(\lambda^*)}{\overline{\lambda}-\lambda^*} \\ &= \triangle c^T\left(\frac{\underline{x}+x^{(1)}}{2}\right) + \triangle b^T\left(\frac{\underline{y}+y^{(1)}}{2}\right). \end{aligned} \qquad (20)$$

Since $(x^{(1)},y^{(1)},s^{(1)})$ is an arbitrary optimal solution at λ^* and $\phi'_+(\lambda^*)$ is independent of the optimal solution choice at λ^*, one may choose $(x^{(1)},y^{(1)},s^{(1)}) = (x,y,s)$ and $(x^{(1)},y^{(1)},s^{(1)}) = (\underline{x},y,s)$. From (20) we get

$$\begin{aligned}\phi'_+(\lambda^*) &= \triangle c^T\left(\frac{\underline{x}+x}{2}\right) + \triangle b^T\left(\frac{\underline{y}+y}{2}\right) \\ &= \triangle c^T\left(\frac{\underline{x}+\underline{x}}{2}\right) + \triangle b^T\left(\frac{\underline{y}+y}{2}\right). \end{aligned} \qquad (21)$$

Equation (21) reduces to $\triangle c^T(\frac{\underline{x}+x}{2}) = \triangle c^T \underline{x}$ from which it follows that $\triangle c^T x = \triangle c^T \underline{x}$. Furthermore, let us consider $(x^{(1)},y^{(1)},s^{(1)}) = (x,y,s)$ and $(x^{(1)},y^{(1)},s^{(1)}) = (x,\underline{y},\underline{s})$. From (20) we obtain $\triangle b^T y = \triangle b^T \underline{y}$.

Now, since both (x,y,s) and $(\underline{x},\underline{y},\underline{s})$ are optimal solutions in $\mathcal{QP}^*_{\lambda^*} \times \mathcal{QD}^*_{\lambda^*}$, it holds that $(c+\lambda^*\triangle c)^T x = (c+\lambda^*\triangle c)^T \underline{x}$ and $(b+\lambda^*\triangle b)^T y = (b+\lambda^*\triangle b)^T \underline{y}$ (see e.g., Dorn (1960)). Consequently, it follows from (17) that $c^T x = c^T \underline{x}$ and $b^T y = b^T \underline{y}$.

As a result we can establish that

$$\begin{aligned} x^T\overline{s} &= x^T(c + \overline{\lambda}\triangle c + Q\overline{x} - A^T\overline{y}) \\ &= c^T x + \overline{\lambda}\triangle c^T x + x^T Q\overline{x} - (b+\lambda^*\triangle b)^T\overline{y} \\ &= c^T\underline{x} + \overline{\lambda}\triangle c^T\underline{x} + \underline{x}^T Q\overline{x} - (A\underline{x})^T\overline{y} \\ &= \underline{x}^T(c + \overline{\lambda}\triangle c + Q\overline{x} - A^T\overline{y}) = \underline{x}^T\overline{s} = 0, \end{aligned} \qquad (22)$$

and

$$\begin{aligned}\overline{x}^T s &= \overline{x}^T(c+\lambda^*\triangle c+Qx-A^Ty)\\ &= \overline{x}^T(c+\lambda^*\triangle c+Qx)-b^Ty-\overline{\lambda}\triangle b^Ty\\ &= \overline{x}^T(c+\lambda^*\triangle c+Q\underline{x})-b^T\underline{y}-\overline{\lambda}\triangle b^T\underline{y}\\ &= \overline{x}^T(c+\lambda^*\triangle c+Q\underline{x}-A^T\underline{y})\\ &= \overline{x}^T\underline{s}=0.\end{aligned} \tag{23}$$

For $\theta\in(0,1)$ and $\tilde{\lambda}=(1-\theta)\lambda^*+\theta\overline{\lambda}$, let us consider

$$\begin{aligned}\tilde{x}&=(1-\theta)x+\theta\overline{x},\\ \tilde{y}&=(1-\theta)y+\theta\overline{y},\\ \tilde{s}&=(1-\theta)s+\theta\overline{s}.\end{aligned} \tag{24}$$

Utilizing equations (24) and the complementarity properties (22) and (23), we obtain that $\tilde{x}$ and $(\tilde{x},\tilde{y},\tilde{s})$ are feasible and complementary, and thus optimal solutions of $(QP_{\tilde{\lambda}})$ and $(QD_{\tilde{\lambda}})$, respectively. Noting that $(\overline{\mathcal{B}},\overline{\mathcal{N}},\overline{\mathcal{T}})$ is the optimal partition at $(\tilde{x},\tilde{y},\tilde{s})$, it follows from (24) that $x_{\overline{\mathcal{B}}}\geq 0,\ x_{\overline{\mathcal{N}}}=0,\ x_{\overline{\mathcal{T}}}=0$ and $s_{\overline{\mathcal{B}}}=0,\ s_{\overline{\mathcal{N}}}\geq 0,\ s_{\overline{\mathcal{T}}}=0$. Then we can conclude that $\sigma(x)\subseteq\overline{\mathcal{B}}$ and $\sigma(s)\subseteq\overline{\mathcal{N}}$.

The next theorem presents two auxiliary linear optimization problems to calculate the left and right second order derivatives of $\phi(\lambda)$ and also gives a general result concerning the transition points of the optimal value function. Problem (16) can be used for finding optimal solutions of problems (14) and (15).

Theorem 13 *Let $\lambda\in\Lambda$, and x^* be an optimal solution of (QP_λ). Further, let (x^*,y^*,s^*) be an optimal solution of (QD_λ). Then, the left and right second order derivatives $\phi''_-(\lambda)$ and $\phi''_+(\lambda)$ are*

$$\begin{aligned}\phi''_-(\lambda)=\min_{\xi,\varrho,\mu,\eta,\rho,\delta}\{&\triangle c^T\xi : A\xi=\triangle b,\\ &\xi+\varrho+\mu x^*=0,\ \varrho_{\sigma(s^-)}\geq 0,\varrho_{\sigma(x^-)}=0,\\ &A^T\eta+\rho-Q\xi+\delta s^*=\triangle c,\\ &\rho_{\sigma(s^-)}\geq 0,\ \rho_{\sigma(x^-)}=0\}\end{aligned}$$

$$\begin{aligned}+\max_{\xi,\varrho,\mu,\eta,\rho,\delta}\{&\triangle b^T\eta : A\xi=\triangle b,\\ &\xi+\varrho+\mu x^*=0,\varrho_{\sigma(s^-)}\geq 0,\varrho_{\sigma(x^-)}=0,\\ &A^T\eta+\rho-Q\xi+\delta s^*=\triangle c,\\ &\rho_{\sigma(s^-)}\geq 0,\ \rho_{\sigma(x^-)}=0\},\end{aligned}$$

where (x^-,y^-,s^-) is a strictly complementary optimal solution of (14), and

$$\begin{aligned}\phi''_+(\lambda)=\max_{\xi,\varrho,\mu,\eta,\rho,\delta}\{&\triangle c^T\xi : A\xi=\triangle b,\\ &\xi+\varrho+\mu x^*=0,\ \varrho_{\sigma(s^+)}\geq 0,\varrho_{\sigma(x^+)}=0,\\ &A^T\eta+\rho-Q\xi+\delta s^*=\triangle c,\\ &\rho_{\sigma(s^+)}\geq 0,\ \rho_{\sigma(x^+)}=0\}\\ +\min_{\xi,\varrho,\mu,\eta,\rho,\delta}\{&\triangle b^T\eta : A\xi=\triangle b,\\ &\xi+\varrho+\mu x^*=0,\ \varrho_{\sigma(s^+)}\geq 0,\varrho_{\sigma(x^+)}=0,\\ &A^T\eta+\rho-Q\xi+\delta s^*=\triangle c,\\ &\rho_{\sigma(s^+)}\geq 0,\ \rho_{\sigma(x^+)}=0\},\end{aligned}$$

where (x^+,y^+,s^+) is a strictly complementary optimal solution of (15).

Proof: The proof follows by using a similar pattern of reasoning as Lemma IV.61 in Roos, Terlaky and Vial (2006) for the linear problems (14) and (15).

The following theorem summarizes the results we got so far. It is a direct consequence of Theorem 4.1 in Yildirim (2004) (equivalence of (i) and (ii)), the definition of a transition point (equivalence of (ii) and (iii)), and Corollary 4 and Lemma 12 (equivalence of (iii) and (iv)). The proof is identical to the proof of Theorem 3.10 in Berkelaar et al. (1997) and it also shows that in adjacent subintervals $\phi(\lambda)$ is defined by different quadratic functions.

Theorem 14 *The following statements are equivalent:*

(i) $D_\pi=\emptyset$*;*
(ii) $\Lambda(\pi)=\{\lambda^*\}$*;*
(iii) λ^* *is a transition point;*
(iv) ϕ' *or* ϕ'' *is discontinuous at* λ^**.*

By solving an auxiliary self-dual quadratic optimization problem one can obtain the optimal partition in the neighboring invariancy interval. The result is given by the next theorem.

Theorem 15 *Let λ^* be a transition point of the optimal value function. Let (x^*,y^*,s^*) be an optimal solution of (15) for λ^*. Let us assume that the (open) invariancy interval to the right of λ^* contains $\overline{\lambda}$ with optimal partition $\overline{\pi}=(\overline{\mathcal{B}},\overline{\mathcal{N}},\overline{\mathcal{T}})$. Define $T=\overline{\sigma}(x^*,s^*)=\{1,2,\ldots,n\}\backslash(\sigma(x^*)\cup\sigma(s^*))$. Consider the following self-dual quadratic problem*

$$\begin{aligned}\min_{\xi,\rho,\eta}\ \{&-\triangle b^T\eta+\triangle c^T\xi+\xi^TQ\xi : A\xi=\triangle b,\\ &A^T\eta+\rho-Q\xi=\triangle c,\\ &\xi_{\sigma(s^*)}=0,\ \rho_{\sigma(x^*)}=0,\xi_{\overline{\sigma}(x^*,s^*)}\geq 0,\\ &\rho_{\overline{\sigma}(x^*,s^*)}\geq 0\},\end{aligned} \tag{25}$$

and let (ξ^*, η^*, ρ^*) be a maximally complementary solution of (25). Then, $\overline{\mathcal{B}} = \sigma(x^*) \cup \sigma(\xi^*)$, $\overline{\mathcal{N}} = \sigma(s^*) \cup \sigma(\rho^*)$ and $\overline{\mathcal{T}} = \{1, \ldots, n\} \setminus (\overline{\mathcal{B}} \cup \overline{\mathcal{N}})$.

Proof: For any feasible solution of (25) we have

$$\begin{aligned} &- \triangle b^T \eta + \triangle c^T \xi + \xi^T Q \xi \\ &\quad = \xi^T (Q\xi - A^T \eta + \triangle c) = \xi^T \rho = \xi_T^T \rho_T \geq 0. \end{aligned}$$

The dual of (25) is

$$\begin{aligned} \max_{\delta,\xi,\gamma,\zeta} \{ \, & \triangle b^T \delta - \triangle c^T \zeta - \xi^T Q \xi : \\ & A\zeta = \triangle b, A^T \delta + \gamma + Q\zeta - 2Q\xi = \triangle c, \\ & \gamma_{\sigma(x^*)} = 0, \zeta_{\sigma(s^*)} = 0, \gamma_T \geq 0, \zeta_T \geq 0 \}. \end{aligned}$$

For a feasible solution it holds

$$\begin{aligned} &\triangle b^T \delta - \triangle c^T \zeta - \xi^T Q \xi \\ &\quad = \delta^T A \zeta - \triangle c^T \zeta - \xi^T Q \xi \\ &\quad = -\zeta^T \gamma - (\zeta - \xi)^T Q (\zeta - \xi) \leq 0. \end{aligned}$$

So, the optimal value of (25) is zero. Let us observe that $(\overline{x}, \overline{y}, \overline{s})$ is a maximally complementary solution at $\overline{\lambda}$ and assign

$$\begin{aligned} \xi = \zeta &= \frac{\overline{x} - x^*}{\overline{\lambda} - \lambda^*}, \\ \eta = \delta &= \frac{\overline{y} - y^*}{\overline{\lambda} - \lambda^*}, \\ \rho = \gamma &= \frac{\overline{s} - s^*}{\overline{\lambda} - \lambda^*}, \end{aligned} \tag{26}$$

that satisfy the first two linear constraints of (25).

Using the fact that by Lemma 12 $\sigma(x^*) \subseteq \overline{\mathcal{B}}$ and $\sigma(s^*) \subseteq \overline{\mathcal{N}}$, it follows that

$$\begin{aligned} \xi_{\sigma(s^*)} &= \frac{\overline{x}_{\sigma(s^*)} - x^*_{\sigma(s^*)}}{\overline{\lambda} - \lambda^*} = 0, \\ \xi_T &= \frac{\overline{x}_T - x^*_T}{\overline{\lambda} - \lambda^*} = \frac{\overline{x}_T}{\overline{\lambda} - \lambda^*} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \rho_{\sigma(x^*)} &= \frac{\overline{s}_{\sigma(x^*)} - s^*_{\sigma(x^*)}}{\overline{\lambda} - \lambda^*} = 0, \\ \rho_T &= \frac{\overline{s}_T - s^*_T}{\overline{\lambda} - \lambda^*} = \frac{\overline{s}_T}{\overline{\lambda} - \lambda^*} \geq 0. \end{aligned}$$

Then, problem (25) is feasible and self-dual.

From the proof of Lemma 12 we have $\overline{x}^T s^* = \overline{s}^T x^* = 0$, implying that (26) is an optimal solution. So, (26) is definitely an optimal solution for (25) as it satisfies all the constraints and gives zero optimal value. On the other hand, since $(\overline{x}, \overline{y}, \overline{s})$ is maximally complementary at $\overline{\lambda}$, we get $\xi_{\sigma(s^*)} = 0$, $\xi_{\sigma(x^*)} > 0$, $\xi_T = \overline{x}_T$, $\rho_{\sigma(x^*)} = 0$, $\rho_{\sigma(s^*)} > 0$ and $\rho_T = \overline{s}_T$ which means that (26) is a maximally complementary solution in (25) as well.

Using (26) and the fact that $\overline{\lambda} > \lambda^*$, we see that $\overline{\mathcal{B}} = \sigma(\overline{x}) = \sigma(x^*) \cup \sigma(\xi)$ and $\overline{\mathcal{N}} = \sigma(\overline{s}) = \sigma(s^*) \cup \sigma(\rho)$. Further, we note that (ξ, η, ρ) defined in (26) is a maximally complimentary solution of (25), and hence $\sigma(\xi) = \sigma(\xi^*)$ and $\sigma(\rho) = \sigma(\rho^*)$. Thus, $\overline{\mathcal{B}} = \sigma(x^*) \cup \sigma(\xi^*)$ follows. Analogous arguments hold for $\overline{\mathcal{N}}$, which completes the proof.

4.3. *Computational Algorithm*

In this subsection we summarize the results in a computational algorithm. This algorithm is capable of finding the transition points; the right first order derivatives of the optimal value function at transition points; and optimal partitions at all transition points and invariancy intervals. Note that the algorithm computes all these quantities to the right from the given initial value λ^*. One can easily outline an analogous algorithm for the transition points to the left from λ^*. It is worthwhile to mention that all the subproblems used in this algorithm can be solved in polynomial time by IPMs.

The implementation of the computational algorithm contains some complications that are worth to mention. The interested reader can find more details about it in Romanko (2004). First, due to numerical errors the determination of the optimal partition and a maximally complementary optimal solution, or the determination of the support set for a given optimal solution is a troublesome task. In contrast with the theoretical results, the numerical solution produced by a CQO solver may not allow to determine the optimal partition or support set with 100% reliability. Introducing a zero tolerance parameter and using some heuristics may improve the situation. For problems with hundreds or thousands of variables, the probability of getting one or more "problematic" coordinates is very high.

Algorithm: Transition Points, First-Order Derivatives of the Optimal Value Function and Optimal Partitions at All Subintervals for CQO

Input:

A nonzero direction of perturbation: $r = (\triangle b, \triangle c)$;
a maximally complementary solution (x^*, y^*, s^*) of (QP_λ) and (QD_λ) for $\lambda = \lambda^*$;
$\pi^0 = (\mathcal{B}^0, \mathcal{N}^0, \mathcal{T}^0)$, where
$\mathcal{B}^0 = \sigma(x^*)$, $\mathcal{N}^0 = \sigma(s^*)$;
$k := 0$; $x^0 := x^*$; $y^0 := y^*$; $s^0 := s^*$;
ready:= false;

while not ready **do**
begin
solve

$$\lambda_k = \max_{\lambda,x,y,s} \{\lambda : Ax - \lambda\triangle b = b,\\ x_{\mathcal{B}^k} \geq 0, x_{\mathcal{N}^k \cup \mathcal{T}^k} = 0,\\ A^T y + s - Qx - \lambda\triangle c = c,\\ s_{\mathcal{N}^k} \geq 0,\ s_{\mathcal{B}^k \cup \mathcal{T}^k} = 0\};$$

if this problem is unbounded: ready:= true; **else**
let $(\lambda_k, x^k, y^k, s^k)$ be an optimal solution;
begin
Let $x^* := x^k$ and $s^* := s^k$;
solve
$\min_{x,y,s}\{\triangle c^T x - \triangle b^T y : (x, y, s) \in \mathcal{QPD}^*_\lambda\}$
if this problem is unbounded: ready:= true; **else**
let (x^k, y^k, s^k) be an optimal solution;
begin
Let $x^* := x^k$ and $s^* := s^k$;
solve

$$\min_{\xi,\rho,\eta}\{-\triangle b^T\eta + \triangle c^T\xi + \xi^T Q\xi : A\xi = \triangle b,\\ A^T\eta + \rho - Q\xi = \triangle c,\ \xi_{\sigma(s^*)} = 0,\\ \rho_{\sigma(x^*)} = 0, \xi_{\overline{\sigma}(x^*,s^*)} \geq 0, \rho_{\overline{\sigma}(x^*,s^*)} \geq 0\};$$

$\mathcal{B}^{k+1} = \sigma(x^*) \cup \sigma(\xi^*), \mathcal{N}^{k+1} = \sigma(s^*) \cup \sigma(\rho^*)$,
$\mathcal{T}^{k+1} = \{1, \ldots, n\} \setminus (\mathcal{B}^{k+1} \cup \mathcal{N}^{k+1})$;
$k := k + 1$;
end
end
end

Wrongly determined tri-partition may lead to an incorrect invariancy interval, if any. The situation can be improved by resolving the problem for another λ parameter value close to the current one. Another possibility to overcome this difficulty in implementation is to resolve the problem with fixed "non-problematic" coordinates in order to obtain a more precise solution for the problematic ones.

Second, incorrectly determined optimal partition or support sets, as well as numerical difficulties, may prevent one of the auxiliary subproblems to be solved. In this case, we can restart the algorithm from a parameter value λ sufficiently close to the current one in order to get the solutions for the whole interval Λ.

Finally, the derivative subproblem (16) is more challenging than it seems. The difficulties here are caused by the fact that we want to solve the derivative subproblem without knowing the optimal partition at the current transition point λ_k, but only by utilizing an optimal solution (x^k, y^k, s^k) that is produced by solving (3). This is actually the reason why we need to have the non-negativity constraints $x_{\sigma(x^k,s^k)} \geq 0$ and $s_{\sigma(x^k,s^k)} \geq 0$, where $\sigma(x^k, s^k) = \{1, 2, \ldots, n\} \setminus (\sigma(x^k) \cup \sigma(s^k))$, in the problem (16) that converts it to:

$$\min_{x,y,s}\{\triangle c^T x - \triangle b^T y : Ax = b + \lambda_k\triangle b,\\ x_{\sigma(x^k)\cup\sigma(x^k,s^k)} \geq 0,\\ x_{\sigma(s^k)} = 0,\ Qx = Qx^k,\\ A^T y + s - Qx = c + \lambda_k\triangle c,\\ s_{\sigma(s^k)\cup\sigma(x^k,s^k)} \geq 0,\ s_{\sigma(x^k)} = 0\}. \tag{27}$$

Presence of these constraints speaks of the fact that we do not actually know to which tri-partition $\mathcal{B}^k$, $\mathcal{N}^k$ or $\mathcal{T}^k$ the indices $\sigma(x^k, s^k)$ will belong. It is the consequence of not having a maximally complementary solution at the current transition point λ_k. This implies that we need to enforce the hidden constraint $(x_{\sigma(x^k,s^k)})_j\,(s_{\sigma(x^k,s^k)})_j = 0 \quad \forall\, j \in \sigma(x^k, s^k)$ for the problem (16). Utilizing the hidden constraints becomes unnecessary, if we know a maximally complementary solution of the parametric problem for λ_k that provides the optimal partition $(\mathcal{B}^k, \mathcal{N}^k, \mathcal{T}^k)$ at this parameter value. Our computational experience shows that if $(x_{\sigma(x,s)})_j > 0$ and $(s_{\sigma(x,s)})_j > 0$ for some j in the optimal solution of (27), then $\mathcal{B} = \sigma(x^k)$ and $\mathcal{N} = \sigma(s^k)$ in that transition point and we exploit this partition while solving (16).

5. Simultaneous Perturbation in Linear Optimization

The case, when perturbation occurs in the objective function vector c or the RHS vector b of an LO problem was extensively studied. A comprehensive survey can be found in the book of Roos, Terlaky and Vial (2006).

Greenberg (2000) has studied simultaneous perturbation of the objective and RHS vectors when the primal and dual LO problems are in canonical form. He only investigated the invariancy interval which includes the current parameter value λ. He proved the convexity of the invariancy interval and established that the optimal value function is quadratic on this interval for the simultaneous perturbation case and it is linear for non-simultaneous perturbation cases. However, for LO problems in canonical form it is necessary to define the optimal partition to separate not only active and inactive variables, but also active and inactive constraints for all optimal solutions. In his approach to identify the optimal value function one needs to know the generalized inverse of the submatrix of A, corresponding to active variables and constraints, in addition to having the optimal solutions at two parameter values.

We start this section by emphasizing the differences in the optimal partitions of the optimal value function in LO and CQO problems and then proceed to specialize our results to the LO case. Let us define the simultaneous perturbation of an LO problem as

$$(LP_\lambda) \quad \min\{(c+\lambda\triangle c)^T x : Ax = b+\lambda\triangle b, x \geq 0\}.$$

Its dual is

$$(LD_\lambda) \ \max\{(b+\lambda\triangle b)^T y : A^T y + s = c+\lambda\triangle c, s \geq 0\}.$$

The LO problem can be derived from the CQO problem by substituting the zero matrix for Q. As a result, vector x does not appear in the constraints of the dual problem, and the set $\mathcal{T}$ in the optimal partition is always empty.

The following theorem shows that to identify an invariancy interval, we don't need to solve problems (2) and (3) as they are formulated for the CQO case. Its proof is based on the fact that the constraints in these problems separate when $Q = 0$, and the proof is left to the reader.

Theorem 16 *Let $\lambda^* \in \Lambda$ be given and let (x^*, y^*, s^*) be a strictly complementary optimal solution of (LP_{λ^*}) and (LD_{λ^*}) with the optimal partition $\pi = (\mathcal{B}, \mathcal{N})$. Then, the left and right extreme points of the interval $\overline{\Lambda}(\pi) = [\lambda_\ell, \lambda_u]$ that contains λ^* are $\lambda_\ell = \max\{\lambda_{P_\ell}, \lambda_{D_\ell}\}$ and $\lambda_u = \min\{\lambda_{P_u}, \lambda_{D_u}\}$, where*

$$\lambda_{P_\ell} = \min_{\lambda,x}\{\lambda : Ax - \lambda\triangle b = b,\ x_\mathcal{B} \geq 0,\ x_\mathcal{N} = 0\},$$
$$\lambda_{P_u} = \max_{\lambda,x}\{\lambda : Ax - \lambda\triangle b = b,\ x_\mathcal{B} \geq 0,\ x_\mathcal{N} = 0\},$$
$$\lambda_{D_\ell} = \min_{\lambda,y,s}\{\lambda : A^T y + s - \lambda\triangle c = c,\ s_\mathcal{N} \geq 0,\ s_\mathcal{B} = 0\},$$
$$\lambda_{D_u} = \max_{\lambda,y,s}\{\lambda : A^T y + s - \lambda\triangle c = c,\ s_\mathcal{N} \geq 0,\ s_\mathcal{B} = 0\}.$$

We also state the following lemma that does not hold for CQO problems.

Lemma 17 *Let λ_ℓ and λ_u be obtained from Theorem 16 and $\lambda_\ell < \lambda_1 < \lambda_2 < \lambda_u$ with $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ being any strictly complementary solutions of (LP_λ) and (LD_λ) corresponding to λ_1 and λ_2, respectively. Then it holds that*

$$\triangle b^T \triangle y = \triangle c^T \triangle x,$$

where $\triangle y = y^{(2)} - y^{(1)}$ and $\triangle x = x^{(2)} - x^{(1)}$.

Proof: Subtracting the constraints of (LP_{λ_1}) from (LP_{λ_2}) and the constraints of (LD_{λ_1}) from (LD_{λ_2}) results in

$$A\triangle x = \triangle\lambda\triangle b, \tag{28}$$
$$A^T\triangle y + \triangle s = \triangle\lambda\triangle c, \tag{29}$$

where $\triangle\lambda = \lambda_2 - \lambda_1$ and $\triangle s = s^{(2)} - s^{(1)}$. Premultiplying (28) by $\triangle y^T$ and (29) by $\triangle x^T$, the result follows from the fact that $\triangle x^T \triangle s = 0$, which completes the proof.

Utilizing Lemma 17 and using the same notation as in (10)–(12), we can state the following theorem that gives explicit expressions for computing the objective value function. The theorem also gives the criteria to determine convexity, concavity and linearity of the objective value function on its subintervals.

Theorem 18 *Let $\lambda_1 < \lambda_2$ and $\pi(\lambda_1) = \pi(\lambda_2) = \pi$, let $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ be strictly complementary optimal solutions of problems (LP_λ) and (LD_λ) at λ_1 and λ_2, respectively. The following statements hold:*

(i) The optimal partition is invariant on (λ_1, λ_2).

(ii) The optimal value function is quadratic on this interval and is given by

$$\begin{aligned}\phi(\lambda) &= (\phi(\lambda_1) - \lambda_1\gamma_1 + \tfrac{1}{2}\lambda_1^2\gamma) \\ &\quad +(\gamma_1 - \lambda_1\gamma)\lambda + \tfrac{1}{2}\gamma\lambda^2 \\ &= \phi(\lambda_1) + \theta\triangle\lambda(\triangle b^T y^{(1)} + \triangle c^T x^{(1)}) \\ &\quad +\theta^2\triangle\lambda\triangle c^T\triangle x \\ &= \phi(\lambda_1) + \theta\triangle\lambda(\triangle b^T y^{(1)} + \triangle c^T x^{(1)}) \\ &\quad +\theta^2\triangle\lambda\triangle b^T\triangle y\end{aligned}$$

(iii) On any subinterval, the objective value function is

- *strictly convex if* $\triangle c^T\triangle x = \triangle b^T\triangle y > 0$,
- *linear if* $\triangle c^T\triangle x = \triangle b^T\triangle y = 0$,
- *strictly concave if* $\triangle c^T\triangle x = \triangle b^T\triangle y < 0$.

Computation of derivatives can be done by solving smaller LO problems than the problems introduced in Theorem 10. The following theorem summarizes these results.

Theorem 19 *For a given* $\lambda \in \Lambda$, *let* (x^*, y^*, s^*) *be a pair of primal-dual optimal solutions of* (LP_λ) *and* (LD_λ). *Then, the left and right first order derivatives of the optimal value function* $\phi(\lambda)$ *at* λ *are*

$$\begin{aligned}\phi'_-(\lambda) &= \min_{y,s}\{\triangle b^T y : A^T y + s = c + \lambda\triangle c, \\ &\qquad s \geq 0,\ s^T x^* = 0\} \\ &\quad + \max_x\{\triangle c^T x : Ax = b + \lambda\triangle b, \\ &\qquad x \geq 0,\ x^T s^* = 0\},\end{aligned}$$

$$\begin{aligned}\phi'_+(\lambda) &= \max_{y,s}\{\triangle b^T y : A^T y + s = c + \lambda\triangle c, \\ &\qquad s \geq 0,\ s^T x^* = 0\} \\ &\quad + \min_x\{\triangle c^T x : Ax = b + \lambda\triangle b, \\ &\qquad x \geq 0,\ x^T s^* = 0\}.\end{aligned}$$

Yildirim (2003) showed that results similar to Theorems 10 and 19 hold for parametric Convex Conic Optimization (CCO) problems.

6. Illustrative Example

Here we present some illustrative numerical results by using the algorithm outlined in Section 4.3. Computations can be performed by using any IPM solver for LO and CQO problems. Let us consider the following CQO problem with $x, c \in \mathbb{R}^5$, $b \in \mathbb{R}^3$, $Q \in \mathbb{R}^{5\times 5}$ being a positive semidefinite symmetric matrix, $A \in \mathbb{R}^{3\times 5}$ with $\text{rank}(A) = 3$. The problem's data are

$$Q = \begin{bmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\quad c = \begin{bmatrix} -16 \\ -20 \\ 0 \\ 0 \\ 0 \end{bmatrix},\quad \triangle c = \begin{bmatrix} 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 2 & 5 & 0 & 0 & 1 \end{bmatrix},\quad b = \begin{bmatrix} 11 \\ 8 \\ 20 \end{bmatrix},\quad \triangle b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

With this data the perturbed CQO instance is

$$\begin{array}{ll} \min & (-16 + 7\lambda)x_1 + (-20 + 6\lambda)x_2 \\ & +2x_1^2 + 2x_1x_2 + \frac{5}{2}x_2^2 \\ \text{s.t.} & 2x_1 + 2x_2 + x_3 = 11 + \lambda \\ & 2x_1 + x_2 + x_4 = 8 + \lambda \\ & 2x_1 + 5x_2 + x_5 = 20 + \lambda \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0. \end{array} \tag{30}$$

The results of our computations are presented in Table 1. The set Λ for the optimal value function $\phi(\lambda)$ is $[-8, +\infty)$. Figure 1 depicts the graph of $\phi(\lambda)$. Transition points and the optimal partitions at each transition point and on the invariancy intervals are identified by solving the problems in Theorems 5 and 15. The optimal value function on the invariancy intervals is computed by using formula (13). Convexity, concavity or linearity of the optimal value function can be determined by the sign of the quadratic term of the optimal value function (see Table 1). As shown in Figure 1, the optimal value function is convex on the first two invariancy intervals, concave on the third and fourth and linear on the last one. The first order derivative does not exists at transition point $\lambda = -5$.

7. A Parametric CQO Model: The DSL Example

One of the recent examples of the use of CQO problems in practice is a model of optimal multi-user spectrum management for Digital Subscriber Lines (DSL)

Table 1

	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{T}$	$\phi(\lambda)$
$\lambda = -8.0$	{3,5}	{1,2,4}	$\emptyset$	
$-8.0 < \lambda < -5.0$	{2,3,5}	{1,4}	$\emptyset$	$68.0\lambda + 8.5\lambda^2$
$\lambda = -5.0$	{2}	{1,3,4,5}	$\emptyset$	
$-5.0 < \lambda < 0.0$	{1,2}	{3,4,5}	$\emptyset$	$-50.0 + 35.5\lambda + 4\lambda^2$
$\lambda = 0.0$	{1,2}	$\emptyset$	{3,4,5}	
$0.0 < \lambda < 1.739$	{1,2,3,4,5}	$\emptyset$	$\emptyset$	$-50.0 + 35.5\lambda - 6.9\lambda^2$
$\lambda = 1.739$	{2,3,4,5}	$\emptyset$	{1}	
$1.739 < \lambda < 3.333$	{2,3,4,5}	{1}	$\emptyset$	$-40.0 + 24.0\lambda - 3.6\lambda^2$
$\lambda = 3.333$	{3,4,5}	{1}	{2}	
$3.333 < \lambda < +\infty$	{3,4,5}	{1,2}	$\emptyset$	0

Transition Points, Invariancy Intervals and Optimal Partitions

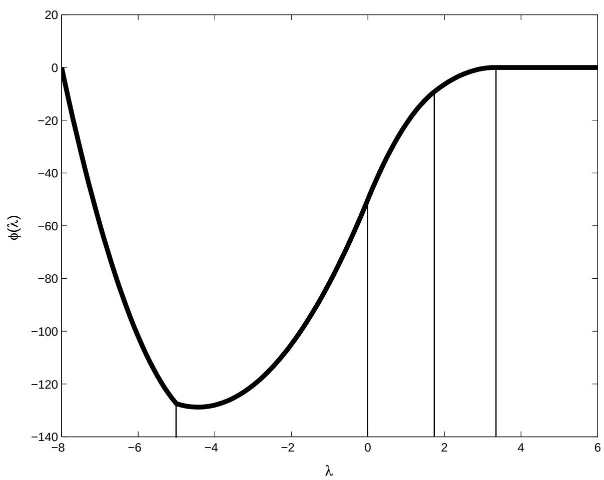

Fig. 1. The Optimal Value Function

that appeared in Yamashita and Luo (2004) as well as in Luo and Pang (2006). Considering the behavior of this model under perturbations, we get a parametric quadratic problem (Romanko 2004). Moreover, the DSL model can have simultaneous perturbation of the coefficients in the objective function and in the right-hand side of the constraints.

Let us consider a situation when M users are connected to one service provider via telephone line (DSL), where M cables are bundled together into the single one. The total bandwidth of the channel is divided into N subcarriers (frequency tones) that are shared by all users. Each user i tries to allocate his total transmission power $P^i_{\max}$ to subcarriers to maximize his data transfer rate

$$\sum_{k=1}^{N} p^i_k = P^i_{\max}.$$

The bundling causes interference between the user lines at each subcarrier $k = 1, \ldots, N$, that is represented by the matrix A_k of cross-talk coefficients. In addition, there is a background noise σ_k at frequency tone k. All elements of matrices A_k are nonnegative with their diagonal elements $a^{ii}_k = 1$. For many practical situations, matrices A_k are positive semidefinite (see Luo and Pang (2006) and subsequent references for more discussion about such cases). For instance with weak cross talk interference scenario, when $0 \le a^{ij}_k \le 1/n$ for all $i \ne j$ and all k, each matrix A_k is strictly diagonally dominant and hence positive definite.

Current DSL systems use fixed power levels. In contrast, allocating each users' total transmission power among the subcarriers "intelligently" may result in higher overall achievable data rates. In noncooperative environment user i allocates his total power $P^i_{\max}$ selfishly across the frequency tones to maximize his own rate. The DSL power allocation problem can be modelled as a multiuser noncooperative game. Nash equilibrium points of the noncooperative rate maximization game correspond to optimal solutions of the following quadratic minimization problem:

$$\begin{aligned} \min & \sum_{k=1}^{N} \sigma_k e^T p_k + \frac{1}{2} \sum_{k=1}^{N} p_k^T A_k p_k \\ s.t. & \sum_{k=1}^{N} p_k^i = P_{\max}^i, \ i = 1, \ldots, M \\ & p_k \geq 0, \ k = 1, \ldots, N, \end{aligned} \tag{31}$$

where $p_k = (p_k^1, \ldots, p_k^M)^T$.

The formulation in (31) provides a convex optimization program that yields, for each user, optimum power allocations across the different subcarriers. However, this formulation assumes that the noise power on each subcarrier is perfectly known *apriori*. Perturbations in the propagation environment due to excessive heat on the line or neighboring bundles may violate this assumption. In order to account for these perturbations one can formulate the problem in (31) as (32):

$$\begin{aligned} \min & \sum_{k=1}^{N} (\sigma_k + \lambda \triangle \sigma_k) e^T p_k + \frac{1}{2} \sum_{k=1}^{N} p_k^T A_k p_k \\ \text{s.t.} & \sum_{k=1}^{N} p_k^i = P_{\max}^i, \ i = 1, \ldots, M \\ & p_k \geq 0, \ k = 1, \ldots, N, \end{aligned} \tag{32}$$

where σ_k now represents the nominal background noise power on the k-th subcarrier and $\Delta\sigma_k$ – the uncertainty in the actual noise power. By varying λ, one can investigate the robustness of the power allocation under the effect of uncertainty in the noise power. In order to mitigate the adverse effect of excessive noise, the i-th user may decide to increase the transmitted power in steps of size $\Delta P_{\max}^i$. Alternatively, if the actual noise is lower than the nominal, the user may decide to decrease the transmitted power. To that end, we can formulate the optimization problem as

$$\begin{aligned} \min & \sum_{k=1}^{N} (\sigma_k + \lambda \triangle \sigma_k) e^T p_k + \frac{1}{2} \sum_{k=1}^{N} p_k^T A_k p_k \\ \text{s.t.} & \sum_{k=1}^{N} p_k^i = P_{\max}^i + \lambda \triangle P_{\max}^i, \\ & i = 1, \ldots, M p_k \geq 0, \ k = 1, \ldots, N, \end{aligned} \tag{33}$$

where the parameter λ is now used to express the uncertainty in noise power as well as power increment to reduce the effect of noise.

8. Conclusions

In this paper we investigated the characteristics of the optimal value function of parametric convex quadratic optimization problems when variation occurs in both the RHS vector of the constraints and the coefficient vector of the objective function's linear term. The rate of variation, represented by the parameter λ, is identical for both perturbation vectors $\triangle b$ and $\triangle c$. We proved that the optimal value function is a continuous piecewise quadratic function on the closed set Λ. Criteria for convexity, concavity or linearity of the optimal value function were derived. Auxiliary linear problems are constructed to find its first and second order left and right derivatives. One of the main results is that the optimal partitions on the left or right neighboring intervals of a given transition point can be determined by solving an auxiliary self-dual quadratic problem. This means that we do not need to guess the length of the invariancy interval to the left or right from the current transition point and should not worry about "missing" short-length invariancy intervals. We already mentioned that all auxiliary problems can be solved in polynomial time. Finally, we outlined an algorithm to identify all invariancy intervals and draw the optimal value function. The algorithm is illustrated with a simple problem. In the special cases, when $\triangle c$ or $\triangle b$ is zero, our findings specialize to the results of Berkelaar et al. (1997) and Roos, Terlaky and Vial (2006). Simplification of some results to LO problems is given, which coincide with the findings of Greenberg (2000).

The most famous application of the CQO sensitivity analysis is the mean-variance portfolio optimization problem introduced by Markowitz (1956). The method presented in our paper allows to analyze not only the original Markowitz model, but also some of its extensions. The link we make to the portfolio problem is based on the tradeoff formulation (see e.g., Steinbach 2001) with the risk aversion parameter λ in the objective function. One possible extension of the tradeoff formulation that results in the simultaneous perturbation model of type (QP_λ) is when the investors's risk aversion parameter λ influences not only risk-return preferences, but also budget constraints. However, we have to stress that simultaneous perturbation in CQO is not solely restricted to portfolio models. There are numerous applications in various engineering areas as we illustrated by the practical example of the adaptive multiuser power allocation for Digital Subscriber Lines.

As some encouraging results already exist for parametric Convex Conic Optimization (CCO), we would like to look at the possibility of extending our algorithm to CCO case. As the content of the previous sentence suggest, our further research directions also include generalizing the analysis of this paper to Second-Order Cone Optimization problems and exploring its applications to financial models.

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References

ADLER, I., AND R. MONTEIRO. 1992. A geometric view of parametric linear programming. *Algorithmica* **8**, 161–176.

BERKELAAR, A. B., B. JANSEN, C. ROOS, AND T. TERLAKY. 1997. An interior point approach to parametric convex quadratic programming. *Working paper*, Erasmus University Rotterdam, Rotterdam, The Netherlands.

BERKELAAR, A. B., C. ROOS, AND T. TERLAKY. 1997. The optimal set and optimal partition approach to linear and quadratic programming. Chapter 6 in *Advances in Sensitivity Analysis and Parametric Programming*, T. Gal and H. J. Greenberg, eds., Kluwer Academic Publishers, Boston, USA.

BOOT, J. C. G. 1964. *Quadratic Programming: Algorithms, Anomalies and Applications*. North-Holland Publishing Company, Amsterdam, The Netherlands.

DORN, W. S. 1960. Duality in quadratic programming. *Quarterly of Applied Mathematics* **18**, 155–162.

GREENBERG, H. J. 2000. Simultaneous primal-dual right-hand-side sensitivity analysis from a strictly complementary solution of a linear program. *SIAM Journal of Optimization* **10**, 427–442.

GÜLER, O., AND Y. YE. 1993. Convergence behavior of interior-point algorithms. *Mathematical Programming* **60**, 215–228.

JANSEN, B., J. J. DE JONG, C. ROOS, AND T. TERLAKY. 1997. Sensitivity analysis in linear programming: just be careful! *European Journal of Operational Research* **101**, 15–28.

LUO, Z.-Q., AND J.-S. PANG. 2006. Analysis of iterative waterfilling algorithm for multiuser power control in Digital Subscriber Lines. *EURASIP Journal on Applied Signal Processing* **2006**, 1–10.

MARKOWITZ, H. M. 1956. The optimization of a quadratic function subject to linear constraints. *Naval Research Logistics Quarterly* **3**, 111–133.

MCLINDEN, L. 1980. An analogue of Moreau's proximation theorem, with application to the nonlinear complementary problem. *Pacific Journal of Mathematics* **88**, 101–161.

MURTY, K. G. 1983. *Linear Programming*. John Wiley & Sons, New York, USA.

ROMANKO, O. 2004. *An Interior Point Approach to Quadratic and Parametric Quadratic Optimization.* M. Sc. Thesis, Advanced Optimization Laboratory, Department of Computing and Software, McMaster University, Hamilton, Canada.

STEINBACH M. 2001. Markowitz revisited: mean-variance models in financial portfolio analysis. *SIAM Review* **43**, 31–85.

ROOS, C., T. TERLAKY, AND J.-PH. VIAL. 2006. *Interior Point Methods for Linear Optimization*, Springer Science+Business Media, New York, USA.

YAMASHITA, N., AND Z.-Q. LUO. 2004. A nonlinear complementarity approach to multiuser power control for digital subscriber lines. *Optimization Methods and Software* **19**, 633–652.

YILDIRIM, E. A. 2004. A unifying optimal partition approach to sensitivity analysis in conic optimization. *Journal of Optimization Theory and Applications* **122**, 405–423.

Appendix

Theorem 4.1 Let $\pi = \pi(\lambda^*) = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ denote the optimal partition for some λ^* and (x^*, y^*, s^*) denote an associated maximally complementary solution at λ^*. Then,

(i) $\Lambda(\pi) = \{\lambda^*\}$ if and only if $D_\pi = \emptyset$;

(ii) $\Lambda(\pi)$ is an open interval if and only if $D_\pi \neq \emptyset$;

(iii) $\mathcal{O}(\pi) = \Lambda(\pi)$ and $cl\,\mathcal{O}(\pi) = cl\,\Lambda(\pi) = \overline{\Lambda}(\pi)$;

(iv) $\overline{\mathcal{S}}_\lambda(\pi) = \{(x, y, s) : x \in \mathcal{QP}^*_\lambda, (x, y, s) \in \mathcal{QD}^*_\lambda\}$ for all $\lambda \in \Lambda(\pi)$.

Proof: First let us recall the characteristics of a maximally complementary solution. Any maximally complementary solution (x^*, y^*, s^*) associated with a given λ^* satisfies $Ax^* = b + \lambda^* \triangle b,\ A^T y^* + s^* - Qx^* = c + \lambda^* \triangle c$, $x^*_{\mathcal{B}} > 0,\ x^*_{\mathcal{N}\cup\mathcal{T}} = 0$, $s^*_{\mathcal{N}} > 0$ and $s^*_{\mathcal{B}\cup\mathcal{T}} = 0$. Let $(\triangle x, \triangle y, \triangle s) \in D_\pi$, and define

$$\overline{x} = x^* + (\overline{\lambda} - \lambda^*)\triangle x, \tag{34}$$

$$\overline{y} = y^* + (\overline{\lambda} - \lambda^*)\triangle y, \tag{35}$$

$$\overline{s} = s^* + (\overline{\lambda} - \lambda^*)\triangle s. \tag{36}$$

If $\overline{\lambda}$ is in an ϵ-neighborhood of λ^* for small enough ϵ, then

$$\begin{aligned} A\overline{x} &= b + \overline{\lambda}\triangle b, \\ A^T\overline{y} + \overline{s} - Q\overline{x} &= c + \overline{\lambda}\triangle c, \\ \overline{x}_{\mathcal{N}\cup\mathcal{T}} &= 0, \\ \overline{s}_{\mathcal{B}\cup\mathcal{T}} &= 0, \\ \overline{x}_{\mathcal{B}} > 0, &\quad \overline{s}_{\mathcal{N}} > 0. \end{aligned} \tag{37}$$

(i) $[\Rightarrow]$: Let $\Lambda(\pi) = \{\lambda^*\}$, and assume to the contrary that D_π is not empty. Then, there exists $(\triangle x, \triangle y, \triangle s)$ such that $A\triangle x = \triangle b$ and $A^T\triangle y + \triangle s - Q\triangle x = \triangle c$ with $\triangle x_{\mathcal{N}\cup\mathcal{T}} = 0$ and $\triangle s_{\mathcal{B}\cup\mathcal{T}} = 0$. Let (x^*, y^*, s^*) be a maximally complementary solution associated with λ^*, i.e., $Ax^* = b + \lambda^*\triangle b$, $A^Ty^* + s^* - Qx^* = c + \lambda^*\triangle c$, $x^*_{\mathcal{N}\cup\mathcal{T}} = 0$, $s^*_{\mathcal{B}\cup\mathcal{T}} = 0$, $x^*_{\mathcal{B}} > 0$ and $s^*_{\mathcal{N}} > 0$. Let $(\overline{x}, \overline{y}, \overline{s})$ be defined by (34)–(36). From (37) one can conclude that $\overline{\lambda} \in \Lambda(\pi)$, what contradicts to the assumption $\Lambda(\pi) = \{\lambda^*\}$.

(i) $[\Leftarrow]$: Let $D_\pi = \emptyset$, and suppose to the contrary that $\overline{\lambda}, \lambda^* \in \Lambda(\pi)$, with $\overline{\lambda} \neq \lambda^*$ and $(\overline{x}, \overline{y}, \overline{s})$ is a maximally complementary solution at $\overline{\lambda}$. Thus, from (34)–(36) we can compute $(\triangle x, \triangle y, \triangle s)$ and conclude that $(\triangle x, \triangle y, \triangle s) \in D_\pi$. This contradicts to the fact that $D_\pi = \emptyset$ and thus $\Lambda(\pi) = \{\lambda^*\}$.

(ii) $[\Rightarrow]$: Let $\lambda^* \in \Lambda(\pi)$. Then, there is a maximally complementary solution (x^*, y^*, s^*) at λ^*. Moreover, since $\Lambda(\pi)$ is an open interval, there exists a $\overline{\lambda}$ in an ϵ-neighborhood of λ^* with $\overline{\lambda} \neq \lambda^*$ and $\overline{\lambda} \in \Lambda(\pi)$. Let $(\overline{x}, \overline{y}, \overline{s})$ denote a maximally complementary solution at $\overline{\lambda}$. From (34)–(36), we can compute $(\triangle x, \triangle y, \triangle s)$ and conclude that $(\triangle x, \triangle y, \triangle s) \in D_\pi \neq \emptyset$.

(ii) $[\Leftarrow]$: Suppose that D_π is non-empty. Then, there exists $(\triangle x, \triangle y, \triangle s)$ such that $A\triangle x = \triangle b$, $A^T\triangle y + \triangle s - Q\triangle x = \triangle c$, $\triangle x_{\mathcal{N}\cup\mathcal{T}} = 0$ and $\triangle s_{\mathcal{B}\cup\mathcal{T}} = 0$. On the other hand, a maximally complementary solution (x^*, y^*, s^*) at λ^* exists such that $Ax^* = b + \lambda^*\triangle b$, $A^Ty^* + s^* - Qx^* = c + \lambda^*\triangle c$, $x^*_{\mathcal{N}\cup\mathcal{T}} = 0, s^*_{\mathcal{B}\cup\mathcal{T}} = 0$, $x^*_{\mathcal{B}} > 0$ and $s^*_{\mathcal{N}} \geq 0$. Consider $(\overline{x}, \overline{y}, \overline{s})$ as defined in (34)–(36). For any $\overline{\lambda} \in \mathbb{R}$, $(\overline{x}, \overline{y}, \overline{s})$ satisfies

$$A\overline{x} = b + \overline{\lambda}\triangle b, \quad A^T\overline{y} + \overline{s} - Q\overline{x} = c + \overline{\lambda}\triangle c,$$

and

$$\overline{x}^T\overline{s} = (\overline{\lambda} - \lambda^*)(\triangle x^T s^* + \triangle s^T x^*).$$

From the definitions of π and D_π, one can conclude that $\overline{x}^T\overline{s} = 0$. Thus $(\overline{x}, \overline{y}, \overline{s})$ is a pair of primal-dual optimal solutions of $(QP_{\overline{\lambda}})$ and $(QD_{\overline{\lambda}})$ as long as $\overline{x} \geq 0$ and $\overline{s} \geq 0$, that gives a closed interval around λ^*. Furthermore, for an open interval Λ, $\overline{x}_{\mathcal{B}} > 0$ and $\overline{s}_{\mathcal{N}} > 0$. Let $\lambda' < \lambda^* < \overline{\lambda}$, where $\lambda', \overline{\lambda} \in \Lambda$. If (x', y', s') and $(\overline{x}, \overline{y}, \overline{s})$ are defined by (34)–(36), then $x'_{\mathcal{B}}, \overline{x}_{\mathcal{B}} > 0$, $x'_{\mathcal{B}\cup\mathcal{T}} = \overline{x}_{\mathcal{B}\cup\mathcal{T}} = 0$, $s'_{\mathcal{N}}, \overline{s}_{\mathcal{N}} > 0$, $s'_{\mathcal{N}\cup\mathcal{T}} = \overline{s}_{\mathcal{N}\cup\mathcal{T}} = 0$. To prove that $\overline{\lambda} \in \Lambda(\pi)$, we need to show that $(\overline{x}, \overline{y}, \overline{s})$ is not only optimal for $(QP_{\overline{\lambda}})$ and $(QD_{\overline{\lambda}})$, but also maximally complementary.

Let us assume that the optimal partition $\overline{\pi} = (\overline{\mathcal{B}}, \overline{\mathcal{N}}, \overline{\mathcal{T}})$ at $\overline{\lambda}$ is not identical to π, i.e., there is a solution $(x(\overline{\lambda}), y(\overline{\lambda}), s(\overline{\lambda}))$ such that

$$\begin{aligned} & x_{\mathcal{B}}(\overline{\lambda}) > 0,\ s_{\mathcal{N}}(\overline{\lambda}) > 0, \\ \text{and} \quad & x_{\mathcal{T}}(\overline{\lambda}) + s_{\mathcal{T}}(\overline{\lambda}) \neq 0. \end{aligned} \tag{38}$$

Let us define

$$\begin{aligned} \tilde{x} &= \frac{\overline{\lambda} - \lambda^*}{\overline{\lambda} - \lambda'} x(\overline{\lambda}) + \frac{\lambda^* - \lambda'}{\overline{\lambda} - \lambda'} x', \\ \tilde{y} &= \frac{\overline{\lambda} - \lambda^*}{\overline{\lambda} - \lambda'} y(\overline{\lambda}) + \frac{\lambda^* - \lambda'}{\overline{\lambda} - \lambda'} y', \\ \tilde{s} &= \frac{\overline{\lambda} - \lambda^*}{\overline{\lambda} - \lambda'} s(\overline{\lambda}) + \frac{\lambda^* - \lambda'}{\overline{\lambda} - \lambda'} s'. \end{aligned}$$

By definition $(\tilde{x}, \tilde{y}, \tilde{s})$ is optimal for λ^*, while by (38) it has a positive $\tilde{x}_i + \tilde{s}_i$ coordinate in $\mathcal{T}$, contradicting to the definition of the optimal partition π at λ*.

We still need to show that $\Lambda(\pi)$ is a connected interval. The proof follows the same reasoning as the proof of Lemma 1 and is omitted.

(iii) Let $\lambda \in \mathcal{O}(\pi)$, then by definition $\pi(\lambda) = \pi$, and hence for $\lambda \in \Lambda$ there is a maximally complementary solution (x, y, s) which satisfies $Ax = b + \lambda\triangle b$, $A^Ty + s - Qx = c + \lambda\triangle c$, $x_{\mathcal{N}\cup\mathcal{T}} = 0$, $s_{\mathcal{B}\cup\mathcal{T}} = 0$, $x_{\mathcal{B}} > 0$ and $s_{\mathcal{N}} > 0$, from which we conclude that $\lambda \in \Lambda(\pi)$. Analogously, one can prove that if $\lambda \in \Lambda(\pi)$ then $\lambda \in \mathcal{O}(\pi)$. Consequently, $\mathcal{O}(\pi) = \Lambda(\pi)$ and $cl\mathcal{O}(\pi) = cl\Lambda(\pi)$. Let us prove that $cl\Lambda(\pi) = \overline{\Lambda}(\pi)$. To the contrary, let $\lambda_1 \notin \overline{\Lambda}(\pi)$ but $\lambda_1 \in cl\Lambda(\pi)$. As $\lambda_1 \in cl\Lambda(\pi)$, λ_1 is a transition point and from the proof of Theorem 4.4, there is an optimal solution $(x^{(1)}, y^{(1)}, s^{(1)})$ at λ_1 with the property, $x^{(1)}_{\mathcal{B}} \geq 0$, $x^{(1)}_{\mathcal{N}\cup\mathcal{T}} = 0$, $s^{(1)}_{\mathcal{N}} \geq 0$ and $s^{(1)}_{\mathcal{N}\cup\mathcal{T}} = 0$. Thus, $\lambda_1 \in \overline{\Lambda}(\pi)$ which is a contradiction. The opposite direction can be proved analogously.

(iv) Let $x^* \in \mathcal{QP}^*_\lambda$ and $(x^*, y^*, s^*) \in \mathcal{QD}^*_\lambda$ are arbitrary optimal solutions of (QP_λ) and (QD_λ) for some λ. Denote by $(\mathcal{B}, \mathcal{N}, \mathcal{T})$ the optimal partition for (QP_λ) and (QD_λ). Now, the optimal sets of the problems are given by:

$$\begin{aligned}\mathcal{QP}^*_\lambda &= \{x \,:\, x \in \mathcal{QP}_\lambda,\ x^T s^* = 0,\ Qx = Qx^*\}\\ &= \{x \,:\, x \in \mathcal{QP}_\lambda, x_{\mathcal{N}\cup\mathcal{T}} = 0\},\end{aligned}$$

$$\begin{aligned}\mathcal{QD}^*_\lambda = \{&(x, y, s) \,:\, (x, y, s) \in \mathcal{QD}_\lambda,\\ &s^T x^* = 0,\ Qx = Qx^*\}\\ = \{&(x, y, s) \,:\, (x, y, s) \in \mathcal{QD}_\lambda,\ s_{\mathcal{B}\cup\mathcal{T}} = 0\}.\end{aligned}$$

The feasible sets used above are $\mathcal{QP}_\lambda = \{x \,:\, Ax = b + \lambda\triangle b,\ x \geq 0\}$ and $\mathcal{QD}_\lambda = \{(x, y, s) \,:\, A^T y + s - Qx = c + \lambda\triangle c,\ s \geq 0\}$. Having the property $Qx = Qx^*$ shows that (x, y, s) is an optimal solution for λ as well.

The fact $x_{\mathcal{B}} \geq 0$ directly follows from $x \geq 0$. On the other hand, since (x, y, s) is a primal-dual optimal solution and (x^*, y^*, s^*) can be chosen as a maximally complementary optimal solution implying $\sigma(x) \subseteq \sigma(x^*)$, thus $x_{\mathcal{N}\cup\mathcal{T}} = 0$ follows immediately. Analogous reasoning is valid for $s_{\mathcal{B}\cup\mathcal{T}} = 0$. As $\overline{S}_\lambda(\pi)$ is defined by $\overline{S}_\lambda(\pi) = \{(x, y, s) \,:\, x \in \mathcal{QP}_\lambda,\ (x, y, s) \in \mathcal{QD}_\lambda,\ x_{\mathcal{B}} \geq 0,\ x_{\mathcal{N}\cup\mathcal{T}} = 0,\ s_{\mathcal{N}} \geq 0,\ s_{\mathcal{B}\cup\mathcal{T}} = 0\}$, we get the statement of the theorem.

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